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NECESSARY CONDITIONS FOR ENERGY MINIMIZERS IN THE NONLINEAR THEORY OF SIX-PARAMETER ELASTIC SHELLS*

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Dedicated to the memory of Professsor H. Brezis

Abstract

In the framework of general nonlinear theory of six-parameter shells we derive pointwise necessary conditions for energy minimizers. We consider conservative problems and exploit the property that the second variation of the potential energy is non-negative if an equilibrium state represents an energy minimizer. Then, using variational calculus we derive the relevant Legendre-Hadamard condition in the theory of shells. Finally, we apply the necessary Legendre-Hadamard inequality to several isotropic strain energy functions proposed previously in the literature on shells.

Keywords: nonlinear 6-parameter shells, Legendre-Hadamard condition, energy minimizers, Cosserat shells, strain energy function.
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1 Introduction

In this paper, we consider the general nonlinear theory of 6-parameter (Cosserat) elastic shells. This is a two-dimensional model for curved shells,

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in which any point $\boldsymbol{x}$ of the deformable midsurface is endowed with six degrees of freedom: three for the translation and three for its rotation. Thus, the basic kinematical variables are the deformation vector field $\boldsymbol{m}(\boldsymbol{x})$ and the microrotation tensor field $\boldsymbol{R}(\boldsymbol{x})$. The foundations of this general shell theory have been presented in [1–4], while several applications and further developments can be found, e.g., in [5–10]. An existence theorem under convexity assumptions has been proved in [11,12]. In these works, the shells are assumed to be made of a classical Cauchy material. However, we mention that the kinematical model of 6-parameter shells coincides with the kinematical model of Cosserat shells in which the three-dimensional material is a Cosserat continuum, as considered in [13, 14]. Moreover, the balance equations have the same forms in the two approaches. The difference resides only in the constitutive assumptions, i.e. in the expression of the strain energy density W, which has to be specified for each particular model (either Cauchy or Cosserat material). In this respect, a higher order refined Cosserat shell model has been derived and investigated in [15–19], where the strain energy density W includes terms of order $O(h^5)$ in the thickness h.

The theoretical results established in this paper hold true for a general strain energy function W (i.e., we make no specific constitutive assumptions), so they are valid for general nonlinear models of 6-parameter or Cosserat shells. In this framework, we obtain necessary conditions of Legendre-Hadamard type for equilibrium states which are energy minimizers. Thus, we derive a counterpart in the theory of shells of the necessary conditions for energy minimizers in three-dimensional Cosserat elasticity presented in [20]. We recall that the Legendre-Hadamard condition has an important physical basis, since it implies the reality of propagation speeds (see, e.g., [21, 22] for the classical elasticity, and [23–25] for the micropolar theory). In the calculus of variations the Legendre-Hadamard inequality is also called ellipticity condition, because it expresses that the Euler equations of the associated functional are elliptic (see, e.g., [26, 27]). Moreover, the Legendre-Hadamard inequality is a necessary condition for stability, in the sense that if the Legendre-Hadamard condition is violated in some point $\boldsymbol{x}$, then the configuration fields $\boldsymbol{m}(\boldsymbol{x})$ and $\boldsymbol{R}(\boldsymbol{x})$ do not represent an energy minimizer. The strict form of the Legendre-Hadamard inequality (also called strong ellipticity condition) coincides with the well-known material stability condition in classical elasticity (see, e.g., [28]). We mention that the Legendre-Hadamard conditions for a Cosserat model of fiber-reinforced elastic solids have been established recently in [29–31]. In the framework of micropolar six-parametric shells regarded as two-dimensional generalized continua, the relevant results concerning constitutive inequali-

ties, strong ellipticity condition, and acceleration waves have been presented first in [4, 32, 33].

To derive the Legendre-Hadamard condition for 6-parameter shells we proceed from the property that the second variation of the potential energy is necessarily non-negative if an equilibrium state represents a minimum of the potential energy [28, 34]. In Section 2 we present briefly the kinematical model of 6-parameter shells and the equilibrium equations, together with the natural boundary conditions, derived from the virtual-power statement. In Section 3 we consider conservative problems and compute the first and second variations of the potential energy functional $\mathcal{E}$ at an equilibrium state. Using the non-negativity property of the second variation $\ddot{\mathcal{E}} \geq 0$ at equilibrium, we derive (in Section 4) the necessary conditions for energy minimizers, expressed in the form of integral inequalities. Then, we adapt the method employed in [20, 35] and use complex-valued functions to obtain in Section 5 the relevant Legendre-Hadamard condition for 6-parameter (Cosserat) shells. Finally, in Section 6 we apply the Legendre-Hadamard inequality to the case of isotropic shells and investigate three specific constitutive models with different complexity degrees. As a result, we obtain in each case the conditions imposed on the constitutive coefficients by the Legendre-Hadamard inequality. In particular, in the case of a strain energy density with coupled membrane and bending terms, we obtain necessary conditions for stability which involve both the material constants and the geometric characteristics of the shell (such as thickness and initial curvature of the midsurface).

In this work, we employ usual notation and conventions, such as the summation convention for diagonally repeated indices. The Latin indices $i, j, \ldots$ range over the set $\{1, 2, 3\}$, while the Greek subscripts and superscripts $\alpha, \beta, \ldots$ range over $\{1, 2\}$. For an arbitrary second order tensor $\boldsymbol{T}$, we denote by $\mathrm{sym}(\boldsymbol{T})$ its symmetric part, $\mathrm{skw}(\boldsymbol{T})$ is the skew part, and $\|\boldsymbol{T}\| = \sqrt{\boldsymbol{T} \cdot \boldsymbol{T}}$ is the norm of $\boldsymbol{T}$. The inner product of two second order tensors $\boldsymbol{T}$ and $\boldsymbol{S}$ is given by $\boldsymbol{T} \cdot \boldsymbol{S} = \mathrm{tr}\big(\boldsymbol{T}\boldsymbol{S}^t\big)$. If $\boldsymbol{\mathcal{M}}$ is a fourth order tensor, then we designate by $\boldsymbol{\mathcal{M}}[\boldsymbol{T}]$ the second order tensor resulting from the linear action of $\boldsymbol{\mathcal{M}}$ on $\boldsymbol{T}$ [21]. For a scalar function $W(\boldsymbol{T}, \boldsymbol{S})$, we denote the partial derivatives with respect to $\boldsymbol{T}$ and $\boldsymbol{S}$ by $W_{\boldsymbol{T}}$ and $W_{\boldsymbol{S}}$, respectively, which are second order tensors. The second partial derivatives of W are the fourth order tensors $W_{\boldsymbol{TT}}$, $W_{\boldsymbol{TS}}$, $W_{\boldsymbol{ST}}$ and $W_{\boldsymbol{SS}}$. Further notations will be introduced in the text, as they appear in the developments.

2 Kinematical model and equilibrium equations

Let $\mathcal{K}$ be the three-dimensional reference configuration of a shell-like body, in which the position vector is denoted by $\boldsymbol{X}(\theta^1,\theta^2,\theta^3)$, the midsurface by Ω and the thickness by h. Here, (θ^i) represent convected curvilinear coordinates $(i=1,2,3)$, where $\theta^3=\zeta$ stands for the thickness coordinate. Thus, we have

$$\boldsymbol{X}(\theta^1,\theta^2,\zeta)=\boldsymbol{x}(\theta^1,\theta^2)+\zeta\,\boldsymbol{n}(\theta^1,\theta^2),\quad (\theta^1,\theta^2)\in\omega\subset\mathbb{R}^2,\quad \zeta\in\Big(-\frac{h}{2},\frac{h}{2}\Big),\tag{1}$$

where $\boldsymbol{x}(\theta^1,\theta^2)$ is the position vector of points on the midsurface Ω and $\boldsymbol{n}(\theta^1,\theta^2)$ is the unit normal to Ω. The domain ω is assumed to be a bounded open connected domain with Lipschitz boundary $\partial\omega$ in the plane $\mathbb{R}^2$. Let $\boldsymbol{a}_\alpha$ and $\boldsymbol{a}^\alpha$ be the covariant and contravariant basis vectors in the tangent plane, respectively, such that

$$\boldsymbol{a}_\alpha=\frac{\partial\boldsymbol{x}}{\partial\theta^\alpha}\,,\qquad \boldsymbol{a}^\alpha\cdot\boldsymbol{a}_\beta=\delta^\alpha_\beta\,,\qquad \boldsymbol{n}=\frac{\boldsymbol{a}_1\times\boldsymbol{a}_2}{\|\boldsymbol{a}_1\times\boldsymbol{a}_2\|}\,,\tag{2}$$

where δ^α_β is the Kronecker symbol $(\alpha,\beta=1,2)$. We also denote with $\boldsymbol{a}_3=\boldsymbol{a}^3=\boldsymbol{n}$ the unit normal vector. Consider the tensor $\mathbb{1}$ given by

$$\mathbb{1}=\boldsymbol{a}_\alpha\otimes\boldsymbol{a}^\alpha=a_{\alpha\beta}\boldsymbol{a}^\alpha\otimes\boldsymbol{a}^\beta=a^{\alpha\beta}\boldsymbol{a}_\alpha\otimes\boldsymbol{a}_\beta\,,\qquad \text{with}\quad a_{\alpha\beta}=\boldsymbol{a}_\alpha\cdot\boldsymbol{a}_\beta\,,\tag{3}$$

and designate by a the determinant $a(\theta^1,\theta^2)=\det\big(a_{\alpha\beta}\big)_{2\times2}>0$. In what follows, we employ the surface gradient ∇_s on the midsurface Ω, which is given by

$$\nabla_s\boldsymbol{f}\;=\;\frac{\partial\boldsymbol{f}}{\partial\theta^\alpha}\otimes\boldsymbol{a}^\alpha\qquad\text{for any field }\ \boldsymbol{f}.\tag{4}$$

Thus, we have $\mathbb{1}=\nabla_s\boldsymbol{x}$, and $\mathbb{1}$ is also called the first fundamental tensor of the surface Ω.

The shell in its reference configuration is described by the position vector $\boldsymbol{x}(\theta^1,\theta^2)$ and the initial microrotation tensor $\boldsymbol{Q}_0(\theta^1,\theta^2)$. Let $\{\boldsymbol{d}^0_1,\boldsymbol{d}^0_2,\boldsymbol{d}^0_3\}$ denote the three orthonormal directors which determine the microrotation tensor $\boldsymbol{Q}_0$ in the reference configuration $\mathcal{K}$. Also, let $\{\boldsymbol{d}_1,\boldsymbol{d}_2,\boldsymbol{d}_3\}$ be the orthonormal triad of directors attached to the deformed configuration $\mathcal{K}_c$. Then, the deformation of the shell is characterized by two fields: the deformation vector $\boldsymbol{m}(\boldsymbol{x})$ and the microrotation tensor $\boldsymbol{R}(\boldsymbol{x})=\boldsymbol{d}_i\otimes\boldsymbol{d}^0_i$. The deformation and microrotation fields $\boldsymbol{m}$ and $\boldsymbol{R}$ are regarded as being independent. Note that these fields may also depend on time, but since this dependence is not important in our development it will not be made explicit.

The nonlinear strain measures for 6-parameter (Cosserat) elastic shells (see, e.g., [2,3,6,12,18]) are given by

$$\boldsymbol{E} = \boldsymbol{R}^T \nabla_s \boldsymbol{m} - \mathbb{1} \qquad \text{(the } \textit{shell strain tensor}\text{)} \tag{5}$$

and

$$\boldsymbol{K} = \mathrm{ax}\Big(\boldsymbol{R}^T \frac{\partial \boldsymbol{R}}{\partial \theta^\alpha}\Big) \otimes \boldsymbol{a}^\alpha \qquad \text{(the } \textit{shell bending-curvature tensor}\text{)}. \tag{6}$$

Let us designate by $\boldsymbol{I}$ the identity tensor in the 3-space and by $\boldsymbol{e} = -\boldsymbol{I} \times \boldsymbol{I}$ the third order Ricci permutation tensor (see, e.g., [13]). We also denote by "$:$" the double-dot product (scalar contraction of two indices) defined by

$$\boldsymbol{X} : \boldsymbol{Y} = X^{ijk} Y_{jkr}\, \boldsymbol{a}_i \otimes \boldsymbol{a}^r \qquad \text{and} \qquad \boldsymbol{X} : \boldsymbol{Z} = X^{ijk} Z_{jk}\, \boldsymbol{a}_i\,,$$

for any tensors of the form $\boldsymbol{X} = X^{ijk} \boldsymbol{a}_i \otimes \boldsymbol{a}_j \otimes \boldsymbol{a}_k$, $\boldsymbol{Y} = Y_{ijk} \boldsymbol{a}^i \otimes \boldsymbol{a}^j \otimes \boldsymbol{a}^k$ and $\boldsymbol{Z} = Z_{ij} \boldsymbol{a}^i \otimes \boldsymbol{a}^j$. Then, we can express the axial vector $\mathrm{ax}(\boldsymbol{S})$ of any skew tensor $\boldsymbol{S}$ as the product

$$\mathrm{ax}(\boldsymbol{S}) = -\frac{1}{2}\, \boldsymbol{e} : \boldsymbol{S}. \tag{7}$$

Accordingly, the bending-curvature tensor (6) can be written in the form

$$\boldsymbol{K} = -\frac{1}{2}\, \boldsymbol{e} : \Big(\boldsymbol{R}^T \frac{\partial \boldsymbol{R}}{\partial \theta^\alpha}\Big) \otimes \boldsymbol{a}^\alpha = -\frac{1}{2}\, \boldsymbol{e} : \boldsymbol{R}^T \Big(\frac{\partial \boldsymbol{R}}{\partial \theta^\alpha} \otimes \boldsymbol{a}^\alpha\Big),$$

i.e.,

$$\boldsymbol{K} = -\frac{1}{2}\, \boldsymbol{e} : \boldsymbol{R}^T \nabla_s \boldsymbol{R}. \tag{8}$$

The areal strain energy density for elastic 6-parameter shells has the form

$$W = W(\boldsymbol{E}, \boldsymbol{K}; \boldsymbol{x}), \tag{9}$$

and the total strain energy is given by

$$\mathcal{W} = \int_\Omega W(\boldsymbol{E}, \boldsymbol{K}; \boldsymbol{x})\, \mathrm{da}\,. \tag{10}$$

In this paper we assume that the function W is continuous with respect to $\boldsymbol{x}$ and twice continuously differentiable with respect to $\boldsymbol{E}$ and $\boldsymbol{K}$.

Equilibrium states of the shell are defined as states that satisfy the virtual-power statement

$$\dot{\mathcal{W}} = \mathcal{P}, \tag{11}$$

where $\mathcal{P}$ is the virtual power of the loads. The explicit expression of $\mathcal{P}$ will be deduced below in (22). Let us derive the equilibrium equations and natural boundary conditions for 6-parameter shells from the virtual-power statement (11). Here, superposed dots identify variational derivatives. These are induced by the derivatives with respect to the parameter ϵ, of the one-parameter deformation and microrotation fields $\boldsymbol{m}(\boldsymbol{x};\epsilon)$ and $\boldsymbol{R}(\boldsymbol{x};\epsilon)$ (evaluated at $\epsilon = 0$), where $\boldsymbol{m}(\boldsymbol{x}) = \boldsymbol{m}(\boldsymbol{x};0)$ and $\boldsymbol{R}(\boldsymbol{x}) = \boldsymbol{R}(\boldsymbol{x};0)$ are equilibrium fields. By the chain rule we have

$$\dot{\mathcal{W}} = \int_\Omega \dot{W}(\boldsymbol{E},\boldsymbol{K};\boldsymbol{x})\,\mathrm{d}a, \qquad \text{with} \qquad \dot{W} = W_{\boldsymbol{E}}\cdot\dot{\boldsymbol{E}} + W_{\boldsymbol{K}}\cdot\dot{\boldsymbol{K}}. \tag{12}$$

Let us determine first the variational derivatives $\dot{\boldsymbol{E}}$ and $\dot{\boldsymbol{K}}$. In view of (5) we have

$$\dot{\boldsymbol{E}} = (\boldsymbol{R}^T\nabla_s\boldsymbol{m} - \mathbb{1})^{\cdot} = \boldsymbol{R}^T\nabla_s\dot{\boldsymbol{m}} + \dot{\boldsymbol{R}}^T\nabla_s\boldsymbol{m}\,. \tag{13}$$

If we denote by

$$\boldsymbol{u} = \dot{\boldsymbol{m}}, \qquad \boldsymbol{\Omega} = \dot{\boldsymbol{R}}\boldsymbol{R}^T, \qquad \text{and} \qquad \boldsymbol{\omega} = \mathrm{ax}(\boldsymbol{\Omega}), \tag{14}$$

where $\boldsymbol{\Omega}$ is a skew tensor, then the relation (13) becomes

$$\dot{\boldsymbol{E}} = \boldsymbol{R}^T(\nabla_s\boldsymbol{u} - \boldsymbol{\Omega}\nabla_s\boldsymbol{m}). \tag{15}$$

Further, let us show that

$$\dot{\boldsymbol{K}} = \boldsymbol{R}^T\nabla_s\boldsymbol{\omega}. \tag{16}$$

Indeed, using the representation (8) we get

$$\begin{aligned}\dot{\boldsymbol{K}} &= -\frac{1}{2}\,\boldsymbol{e} : (\boldsymbol{R}^T\nabla_s\boldsymbol{R})^{\cdot} = -\frac{1}{2}\,\boldsymbol{e} : (\dot{\boldsymbol{R}}^T\nabla_s\boldsymbol{R} + \boldsymbol{R}^T\nabla_s\dot{\boldsymbol{R}}) \\ &= -\frac{1}{2}\,\boldsymbol{e} : \big(-\boldsymbol{R}^T\boldsymbol{\Omega}\nabla_s\boldsymbol{R} + \boldsymbol{R}^T\nabla_s(\boldsymbol{\Omega}\boldsymbol{R})\big).\end{aligned} \tag{17}$$

Here, we have

$$\nabla_s(\boldsymbol{\Omega}\boldsymbol{R}) = \frac{\partial(\boldsymbol{\Omega}\boldsymbol{R})}{\partial\theta^\alpha}\otimes\boldsymbol{a}^\alpha = \Big(\frac{\partial\boldsymbol{\Omega}}{\partial\theta^\alpha}\Big)\boldsymbol{R}\otimes\boldsymbol{a}^\alpha + \boldsymbol{\Omega}\nabla_s\boldsymbol{R},$$

so the relation (17) reduces to

$$\begin{aligned}\dot{\boldsymbol{K}} &= -\frac{1}{2}\,\boldsymbol{e} : \Big(\boldsymbol{R}^T\frac{\partial\boldsymbol{\Omega}}{\partial\theta^\alpha}\,\boldsymbol{R}\Big)\otimes\boldsymbol{a}^\alpha = \mathrm{ax}\Big(\boldsymbol{R}^T\frac{\partial\boldsymbol{\Omega}}{\partial\theta^\alpha}\,\boldsymbol{R}\Big)\otimes\boldsymbol{a}^\alpha \\ &= \boldsymbol{R}^T\mathrm{ax}\Big(\frac{\partial\boldsymbol{\Omega}}{\partial\theta^\alpha}\Big)\otimes\boldsymbol{a}^\alpha = \boldsymbol{R}^T\nabla_s\boldsymbol{\omega}.\end{aligned}$$

The equation (16) is proved. Inserting (15) and (16) in $(12)_2$ we obtain

$$\dot{W} \;=\; \boldsymbol{R}\, W_{\boldsymbol{E}} \cdot (\nabla_s \boldsymbol{u} - \boldsymbol{\Omega} \nabla_s \boldsymbol{m}) + \boldsymbol{R}\, W_{\boldsymbol{K}} \cdot \nabla_s \boldsymbol{\omega}. \tag{18}$$

We can transform the product

$$\begin{aligned} \boldsymbol{R}W_{\boldsymbol{E}} \cdot \boldsymbol{\Omega}\nabla_s \boldsymbol{m} \;\; &= \boldsymbol{R}\, W_{\boldsymbol{E}} (\nabla_s \boldsymbol{m})^T \cdot \boldsymbol{\Omega} = \mathrm{skw}\big(\boldsymbol{R}\, W_{\boldsymbol{E}} (\nabla_s \boldsymbol{m})^T\big) \cdot \boldsymbol{\Omega} \\ &= 2\mathrm{ax}\big[\mathrm{skw}\big(\boldsymbol{R}\, W_{\boldsymbol{E}} (\nabla_s \boldsymbol{m})^T\big)\big] \cdot \boldsymbol{\omega} \end{aligned}$$

and, hence, the variational derivative (18) takes the form

$$\dot{W} = \boldsymbol{R}\, W_{\boldsymbol{E}} \cdot \nabla_s \boldsymbol{u} - 2\,\mathrm{ax}\big[\mathrm{skw}\big(\boldsymbol{R}\, W_{\boldsymbol{E}} (\nabla_s \boldsymbol{m})^T\big)\big] \cdot \boldsymbol{\omega} + \boldsymbol{R}\, W_{\boldsymbol{K}} \cdot \nabla_s \boldsymbol{\omega}. \tag{19}$$

Next, we integrate the last equation and use relations of the type

$$\boldsymbol{T} \cdot \nabla_s \boldsymbol{v} = \mathrm{Div}_s(\boldsymbol{T}^T \boldsymbol{v}) - \boldsymbol{v} \cdot \mathrm{Div}_s \boldsymbol{T}\,, \tag{20}$$

where Div_s is the surface divergence given by $\mathrm{Div}_s \boldsymbol{T} = \frac{\partial \boldsymbol{T}}{\partial \theta^\alpha}\, \boldsymbol{a}^\alpha$ and $\mathrm{Div}_s \boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial \theta^\alpha} \cdot \boldsymbol{a}^\alpha$, for any tensor field $\boldsymbol{T}$ and vector field $\boldsymbol{v}$, respectively. Thus, applying the divergence theorem for surfaces (see, e.g., [34, 36]), the virtual-power equation (11) becomes

$$\begin{aligned} -\int_\Omega \Big\{ \mathrm{Div}_s(\boldsymbol{R}W_{\boldsymbol{E}}) \cdot \boldsymbol{u} + \mathrm{Div}_s(\boldsymbol{R}W_{\boldsymbol{K}}) \cdot \boldsymbol{\omega} + 2\mathrm{ax}\big[\mathrm{skw}\big(\boldsymbol{R}\, W_{\boldsymbol{E}} (\nabla_s \boldsymbol{m})^T\big)\big] \cdot \boldsymbol{\omega} \Big\} \mathrm{d}a \\ + \int_{\partial\Omega} \Big[(\boldsymbol{R}W_{\boldsymbol{E}}) \boldsymbol{\nu} \cdot \boldsymbol{u} + (\boldsymbol{R}W_{\boldsymbol{K}}) \boldsymbol{\nu} \cdot \boldsymbol{\omega} \Big] \mathrm{d}\ell = \mathcal{P}, \end{aligned} \tag{21}$$

where $\boldsymbol{\nu}$ is the exterior unit normal vector to the boundary curve $\partial\Omega$, lying in the plane tangent to Ω. We see that the virtual power has the following form

$$\mathcal{P} = \int_\Omega (\boldsymbol{f} \cdot \boldsymbol{u} + \boldsymbol{l} \cdot \boldsymbol{\omega}) \mathrm{d}a + \int_{\partial\Omega} (\boldsymbol{t} \cdot \boldsymbol{u} + \boldsymbol{c} \cdot \boldsymbol{\omega})\, \mathrm{d}\ell\,, \tag{22}$$

where $\boldsymbol{f}$ and $\boldsymbol{l}$ are densities of force and couple acting in Ω, while $\boldsymbol{t}$ and $\boldsymbol{c}$ are densities of force and couple acting on the boundary $\partial\Omega$.

Since the fields $\boldsymbol{u}$ and $\boldsymbol{\omega}$ are independent and arbitrary, we obtain from (21), (22) and the fundamental lemma of calculus of variations the following equilibrium equations

$$\begin{aligned} &\mathrm{Div}_s(\boldsymbol{R}W_{\boldsymbol{E}}) + \boldsymbol{f} = \boldsymbol{0} \qquad \text{and} \\ &\mathrm{Div}_s(\boldsymbol{R}W_{\boldsymbol{K}}) + 2\,\mathrm{ax}\big[\mathrm{skw}\big(\boldsymbol{R}\, W_{\boldsymbol{E}} (\nabla_s \boldsymbol{m})^T\big)\big] + \boldsymbol{l} = \boldsymbol{0} \qquad \text{in } \; \Omega, \end{aligned} \tag{23}$$

and the boundary conditions

$$(\boldsymbol{R}W_{\boldsymbol{E}})\boldsymbol{\nu} = \boldsymbol{t} \quad \text{on } \partial\Omega_t\,, \qquad (\boldsymbol{R}W_{\boldsymbol{K}})\boldsymbol{\nu} = \boldsymbol{c} \quad \text{on } \partial\Omega_c\,, \tag{24}$$

where $\partial\Omega_t$ is the part of the boundary $\partial\Omega$ where position is not assigned, and $\partial\Omega_c$ is the subset of $\partial\Omega$ where microrotation is not assigned. We consider that position is assigned on $\partial\Omega \setminus \partial\Omega_t$ and microrotation is assigned on $\partial\Omega \setminus \partial\Omega_c$. Thus, the variations $\boldsymbol{u}$ and $\boldsymbol{\omega}$ satisfy

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega \setminus \partial\Omega_t \qquad \text{and} \qquad \boldsymbol{\omega} = \boldsymbol{0} \quad \text{on } \partial\Omega \setminus \partial\Omega_c\,. \tag{25}$$

Accordingly, the fields $\boldsymbol{u}$ and $\boldsymbol{\omega}$ are called *admissible* if they fulfill the conditions (25) and have the degree of regularity implied by the foregoing derivation.

Equations (23) and (24) represent the equilibrium conditions for six-parameter (Cosserat) elastic shells. These equations have been presented previously in the literature on shells (see, e.g., [2, 3, 6, 12]) using various notations.

Remark 1. One can write alternatively the equilibrium equation $(23)_2$ using the vector invariant $\big(\,\cdot\,\big)_\times$ (also called Gibbsian cross) instead of the operator $\mathrm{ax}\big(\mathrm{skw}(\cdot)\big)$. Indeed, for any second order tensor $\boldsymbol{T}$, the vector invariant $\boldsymbol{T}_\times$ satisfies the relation

$$\boldsymbol{T}_\times = -2\,\mathrm{ax}(\mathrm{skw}\,\boldsymbol{T})\,. \tag{26}$$

Here, we recall that the vector invariant $\boldsymbol{T}_\times$ is defined for any tensor expressed in the form $\boldsymbol{T} = \sum_{i=1}^{3} \boldsymbol{x}_{(i)} \otimes \boldsymbol{y}_{(i)}$ by the relation (see, e.g., [37])

$$\boldsymbol{T}_\times = \Big(\sum_{i=1}^{3} \boldsymbol{x}_{(i)} \otimes \boldsymbol{y}_{(i)}\Big)_\times = \sum_{i=1}^{3} \boldsymbol{x}_{(i)} \times \boldsymbol{y}_{(i)}\,. \tag{27}$$

Then, the second equilibrium equation $(23)_2$ can be written in the equivalent form

$$\mathrm{Div}_s(\boldsymbol{R}W_{\boldsymbol{K}}) - \big[\boldsymbol{R}\,W_{\boldsymbol{E}}(\nabla_s \boldsymbol{m})^T\big]_\times + \boldsymbol{l} = \boldsymbol{0} \qquad \text{in } \Omega. \tag{28}$$

3 First and second variations of the energy at equilibrium

In this paper, we consider conservative problems. Thus, we assume the existence of a potential energy $\mathcal{E}$ such that

$$\mathcal{E} = \mathcal{W} - \mathcal{L} \qquad \text{(modulo an additive constant)}, \tag{29}$$

where $\mathcal{L}$ is a load potential, whose variational derivative is equal to the virtual power, i.e. $\dot{\mathcal{L}} = \mathcal{P}$. Then, by virtue of (11) we have

$$\dot{\mathcal{E}} = \dot{\mathcal{W}} - \dot{\mathcal{L}} = \dot{\mathcal{W}} - \mathcal{P} = 0.$$

Thus, equilibrium states can be seen as states that render the potential energy stationary, i.e.

$$\dot{\mathcal{E}} = 0 \qquad \text{(for all admissible } \boldsymbol{u}, \boldsymbol{\omega}). \tag{30}$$

This is a restatement of the virtual-power principle (11). Therefore, we see that energy minimizers necessarily satisfy the equilibrium equations (23) and the boundary conditions (24).

In what follows, we confine our attention to dead–load problems. These are conservative problems in which the load potential has the form (modulo an additive constant)

$$\mathcal{L} = \int_\Omega (\boldsymbol{f}\cdot\boldsymbol{m} + \boldsymbol{N}\cdot\boldsymbol{R})\mathrm{d}a + \int_{\partial\Omega_t} \boldsymbol{t}\cdot\boldsymbol{m}\,\mathrm{d}\ell + \int_{\partial\Omega_c} \boldsymbol{M}\cdot\boldsymbol{R}\,\mathrm{d}\ell\,, \tag{31}$$

where $\boldsymbol{f}$, $\boldsymbol{t}$ are assigned configuration–independent vector fields, while $\boldsymbol{N}, \boldsymbol{M}$ are assigned configuration–independent tensor fields. The vector $\boldsymbol{f}$ corresponds to the density of force acting in Ω, and $\boldsymbol{t}$ is as in the boundary condition $(24)_1$. In view of the relations

$$\boldsymbol{N}\cdot\dot{\boldsymbol{R}} = \boldsymbol{N}\boldsymbol{R}^T\cdot\boldsymbol{\Omega} = \mathrm{skw}(\boldsymbol{N}\boldsymbol{R}^T)\cdot\boldsymbol{\Omega} = 2\,\mathrm{ax}\big[\mathrm{skw}(\boldsymbol{N}\boldsymbol{R}^T)\big]\cdot\boldsymbol{\omega}\,,$$

we see that

$$\dot{\mathcal{L}} = \int_\Omega (\boldsymbol{f}\cdot\boldsymbol{u} + \boldsymbol{l}\cdot\boldsymbol{\omega})\mathrm{d}a + \int_{\partial\Omega_t} \boldsymbol{t}\cdot\boldsymbol{u}\,\mathrm{d}\ell + \int_{\partial\Omega_c} \boldsymbol{c}\cdot\boldsymbol{\omega}\,\mathrm{d}\ell\,, \tag{32}$$

where

$$\boldsymbol{l} = 2\,\mathrm{ax}\big[\mathrm{skw}(\boldsymbol{N}\boldsymbol{R}^T)\big] \qquad \text{and} \qquad \boldsymbol{c} = 2\,\mathrm{ax}\big[\mathrm{skw}(\boldsymbol{M}\boldsymbol{R}^T)\big]. \tag{33}$$

Hence, we notice that the couple density $\boldsymbol{l}$ and the couple traction $\boldsymbol{c}$ are configuration–dependent in the dead–load problem. In this respect, it is known that configuration–independent couples are associated with non–conservative problems [38].

In view of the above relations (18), (29) and (32), the first variation of the energy is given by

$$\dot{\mathcal{E}} = \int_\Omega \Big(\boldsymbol{R}\,W_{\boldsymbol{E}}\cdot\nabla_s\boldsymbol{u} + \boldsymbol{R}\,W_{\boldsymbol{K}}\cdot\nabla_s\boldsymbol{\omega} - \boldsymbol{R}\,W_{\boldsymbol{E}}(\nabla_s\boldsymbol{m})^T\cdot\boldsymbol{\Omega} - \boldsymbol{f}\cdot\boldsymbol{u} - \boldsymbol{l}\cdot\boldsymbol{\omega}\Big)\mathrm{d}a$$
$$- \int_{\partial\Omega_t} \boldsymbol{t}\cdot\boldsymbol{u}\,\mathrm{d}\ell - \int_{\partial\Omega_c} \boldsymbol{c}\cdot\boldsymbol{\omega}\,\mathrm{d}\ell\,. \tag{34}$$

The first variation (34) vanishes for all admissible fields $\boldsymbol{u}$ and $\boldsymbol{\omega}$ if and only if $\{\boldsymbol{m}, \boldsymbol{R}\}$ is an equilibrium state.

Let us consider now the second variation of the energy at equilibrium. If the equilibrium state $\{\boldsymbol{m}, \boldsymbol{R}\}$ is an energy minimizer, then it is necessary that the second variation of the energy in $\{\boldsymbol{m}, \boldsymbol{R}\}$ is non-negative, i.e.

$$\ddot{\mathcal{E}}\,[\boldsymbol{m}, \boldsymbol{R}] \,\geq\, 0 \tag{35}$$

for all vector fields $\boldsymbol{u}$ and $\boldsymbol{\omega}$ satisfying the boundary conditions (25).

In order to compute the second variation, we consider again the one parameter family of deformation and microrotation fields $\boldsymbol{m}(\boldsymbol{x};\epsilon)$ and $\boldsymbol{R}(\boldsymbol{x};\epsilon)$ such that $\boldsymbol{m}(\boldsymbol{x}) = \boldsymbol{m}(\boldsymbol{x};0)$ and $\boldsymbol{R}(\boldsymbol{x}) = \boldsymbol{R}(\boldsymbol{x};0)$ are equilibrated. Denoting with a prime $(\cdot)'$ the derivative with respect to ϵ, we can write the variations

$$\boldsymbol{u}(\boldsymbol{x}) = \dot{\boldsymbol{m}}(\boldsymbol{x}) = \boldsymbol{m}'(\boldsymbol{x};0) \qquad \text{and} \qquad \boldsymbol{y}(\boldsymbol{x}) = \ddot{\boldsymbol{m}}(\boldsymbol{x}) = \boldsymbol{m}''(\boldsymbol{x};0). \tag{36}$$

Let us introduce the skew tensor field $\boldsymbol{W}(\boldsymbol{x};\epsilon)$ given by

$$\boldsymbol{W}(\boldsymbol{x};\epsilon) = \boldsymbol{R}'(\boldsymbol{x};\epsilon)\boldsymbol{R}^T(\boldsymbol{x};\epsilon) \tag{37}$$

and define the skew tensors $\boldsymbol{\Omega}(\boldsymbol{x})$ and $\boldsymbol{\Phi}(\boldsymbol{x})$ by

$$\boldsymbol{\Omega}(\boldsymbol{x}) = \boldsymbol{W}(\boldsymbol{x};0) = \dot{\boldsymbol{R}}(\boldsymbol{x})\boldsymbol{R}^T(\boldsymbol{x}) \qquad \text{and} \qquad \boldsymbol{\Phi}(\boldsymbol{x}) = \boldsymbol{W}'(\boldsymbol{x};0) = \dot{\boldsymbol{W}}(\boldsymbol{x}), \tag{38}$$

as well as the axial vectors

$$\boldsymbol{w}(\boldsymbol{x};\epsilon) = \mathrm{ax}\big(\boldsymbol{W}(\boldsymbol{x};\epsilon)\big), \qquad \boldsymbol{\omega}(\boldsymbol{x}) = \mathrm{ax}\big(\boldsymbol{\Omega}(\boldsymbol{x})\big) = \boldsymbol{w}(\boldsymbol{x};0),$$
$$\boldsymbol{\varphi}(\boldsymbol{x}) = \mathrm{ax}\big(\boldsymbol{\Phi}(\boldsymbol{x})\big) = \dot{\boldsymbol{w}}(\boldsymbol{x}).$$

Then, we can write

$$\begin{aligned} \boldsymbol{R}''(\boldsymbol{x};\epsilon) &= \boldsymbol{W}'(\boldsymbol{x};\epsilon)\boldsymbol{R}(\boldsymbol{x};\epsilon) + \boldsymbol{W}(\boldsymbol{x};\epsilon)\boldsymbol{R}'(\boldsymbol{x};\epsilon) \\ &= \boldsymbol{W}'(\boldsymbol{x};\epsilon)\boldsymbol{R}(\boldsymbol{x};\epsilon) + \boldsymbol{W}^2(\boldsymbol{x};\epsilon)\boldsymbol{R}(\boldsymbol{x};\epsilon). \end{aligned}$$

Putting $\epsilon = 0$ in the last relation, we deduce

$$\ddot{\boldsymbol{R}} = \boldsymbol{\Phi}\boldsymbol{R} + \boldsymbol{\Omega}^2\boldsymbol{R}\,. \tag{39}$$

Next, we compute the derivatives of terms appearing in $\mathcal{E}'$ (cf. (34))

$$\big(\boldsymbol{R}W_{\boldsymbol{E}}\cdot\nabla_s\boldsymbol{m}'\big)' = \boldsymbol{R}W_{\boldsymbol{E}}\cdot\nabla_s\boldsymbol{m}'' + \boldsymbol{W}\boldsymbol{R}W_{\boldsymbol{E}}\cdot\nabla_s\boldsymbol{m}' + \boldsymbol{R}\big(W_{\boldsymbol{E}}\big)'\cdot\nabla_s\boldsymbol{m}'\,, \tag{40}$$

and

$$\big(\boldsymbol{R}W_{\boldsymbol{K}}\cdot\nabla_s\boldsymbol{w}\big)' = \boldsymbol{R}W_{\boldsymbol{K}}\cdot\nabla_s\boldsymbol{w}' + \boldsymbol{W}\boldsymbol{R}W_{\boldsymbol{K}}\cdot\nabla_s\boldsymbol{w} + \boldsymbol{R}\big(W_{\boldsymbol{K}}\big)'\cdot\nabla_s\boldsymbol{w}\,, \tag{41}$$

and

$$\big(\boldsymbol{R}W_{\boldsymbol{E}}(\nabla_s\boldsymbol{m})^T\cdot\boldsymbol{W}\big)' = \boldsymbol{R}W_{\boldsymbol{E}}(\nabla_s\boldsymbol{m})^T\cdot\boldsymbol{W}' + \boldsymbol{W}\boldsymbol{R}W_{\boldsymbol{E}}(\nabla_s\boldsymbol{m})^T\cdot\boldsymbol{W} + \boldsymbol{R}W_{\boldsymbol{E}}(\nabla_s\boldsymbol{m}')^T\cdot\boldsymbol{W} + \boldsymbol{R}\big(W_{\boldsymbol{E}}\big)'(\nabla_s\boldsymbol{m})^T\cdot\boldsymbol{W}\,. \tag{42}$$

Also, we have

$$(\boldsymbol{l}\cdot\boldsymbol{w})' = \boldsymbol{l}\cdot\boldsymbol{w}' + 2\,\mathrm{ax}\big[\mathrm{skw}(\boldsymbol{N}\boldsymbol{R}^T)'\big]\cdot\boldsymbol{w} = \boldsymbol{l}\cdot\boldsymbol{w}' + \boldsymbol{N}(\boldsymbol{R}')^T\cdot\boldsymbol{W} = \boldsymbol{l}\cdot\boldsymbol{w}' + \boldsymbol{N}\boldsymbol{R}^T\cdot\boldsymbol{W}^2 \tag{43}$$

and a similar equation holds for $(\boldsymbol{c}\cdot\boldsymbol{w})'$. Using relations (40)-(43) and putting here $\epsilon = 0$, we can write the second variation of the energy at equilibrium in the form

$$\begin{aligned}\ddot{\mathcal{E}} = & \int_\Omega\Big(\boldsymbol{R}\,W_{\boldsymbol{E}}\cdot\nabla_s\boldsymbol{y} + \boldsymbol{R}\,W_{\boldsymbol{K}}\cdot\nabla_s\boldsymbol{\varphi} - \boldsymbol{R}\,W_{\boldsymbol{E}}(\nabla_s\boldsymbol{m})^T\cdot\boldsymbol{\Phi} - \boldsymbol{f}\cdot\boldsymbol{y} - \boldsymbol{l}\cdot\boldsymbol{\varphi}\Big)\mathrm{d}a \\ & -\int_{\partial\Omega_t}\boldsymbol{t}\cdot\boldsymbol{y}\,\mathrm{d}\ell - \int_{\partial\Omega_c}\boldsymbol{c}\cdot\boldsymbol{\varphi}\,\mathrm{d}\ell\ + \int_\Omega\Big(2\boldsymbol{\Omega}\boldsymbol{R}\,W_{\boldsymbol{E}}\cdot\nabla_s\boldsymbol{u} + \boldsymbol{\Omega}\boldsymbol{R}\,W_{\boldsymbol{K}}\cdot\nabla_s\boldsymbol{\omega} \\ & -\boldsymbol{\Omega}\boldsymbol{R}\,W_{\boldsymbol{E}}(\nabla_s\boldsymbol{m})^T\cdot\boldsymbol{\Omega}\Big)\mathrm{d}a + \int_\Omega\Big(\boldsymbol{R}\big(W_{\boldsymbol{E}}\big)^{\boldsymbol{\cdot}}\cdot\nabla_s\boldsymbol{u} + \boldsymbol{R}\big(W_{\boldsymbol{K}}\big)^{\boldsymbol{\cdot}}\cdot\nabla_s\boldsymbol{\omega} \\ & -\boldsymbol{R}\,(W_{\boldsymbol{E}})^{\boldsymbol{\cdot}}(\nabla_s\boldsymbol{m})^T\cdot\boldsymbol{\Omega} - \boldsymbol{N}\boldsymbol{R}^T\cdot\boldsymbol{\Omega}^2\Big)\mathrm{d}a - \int_{\partial\Omega_c}\boldsymbol{M}\boldsymbol{R}^T\cdot\boldsymbol{\Omega}^2\,\mathrm{d}\ell.\end{aligned} \tag{44}$$

Since $\boldsymbol{y} = \boldsymbol{0}$ on $\partial\Omega\setminus\partial\Omega_t$ and $\boldsymbol{\varphi} = \boldsymbol{0}$ on $\partial\Omega\setminus\partial\Omega_c$, we see that the first three integrals in the right-hand side of (44) vanish, by virtue of (30) with (34). Further, inserting here the derivatives

$$\big(W_{\boldsymbol{E}}\big)^{\boldsymbol{\cdot}} = W_{\boldsymbol{EE}}[\dot{\boldsymbol{E}}] + W_{\boldsymbol{EK}}[\dot{\boldsymbol{K}}] \quad \text{and} \quad \big(W_{\boldsymbol{K}}\big)^{\boldsymbol{\cdot}} = W_{\boldsymbol{KE}}[\dot{\boldsymbol{E}}] + W_{\boldsymbol{KK}}[\dot{\boldsymbol{K}}], \tag{45}$$

we obtain the final form of the second variation of the energy at equilibrium

$$\begin{aligned}\ddot{\mathcal{E}} = & \int_\Omega\Big(\boldsymbol{R}^T\nabla_s\boldsymbol{u}\cdot W_{\boldsymbol{EE}}[\boldsymbol{R}^T\nabla_s\boldsymbol{u}] + 2\boldsymbol{R}^T\nabla_s\boldsymbol{u}\cdot W_{\boldsymbol{EK}}[\boldsymbol{R}^T\nabla_s\boldsymbol{\omega}] \\ & +\boldsymbol{R}^T\nabla_s\boldsymbol{\omega}\cdot W_{\boldsymbol{KK}}[\boldsymbol{R}^T\nabla_s\boldsymbol{\omega}]\Big)\mathrm{d}a + \int_\Omega F(\boldsymbol{m},\boldsymbol{R},\boldsymbol{u},\boldsymbol{\Omega})\,\mathrm{d}a - \int_{\partial\Omega_c}\boldsymbol{M}\boldsymbol{R}^T\cdot\boldsymbol{\Omega}^2\mathrm{d}\ell,\end{aligned} \tag{46}$$

where we have denoted by F the expression

$$\begin{aligned}F(\boldsymbol{m},\boldsymbol{R},\boldsymbol{u},\boldsymbol{\Omega}) = & \,2\boldsymbol{\Omega}\boldsymbol{R}\,W_{\boldsymbol{E}}\cdot\nabla_s\boldsymbol{u} + \boldsymbol{\Omega}\boldsymbol{R}\,W_{\boldsymbol{K}}\cdot\nabla_s\boldsymbol{\omega} - \boldsymbol{\Omega}\boldsymbol{R}\,W_{\boldsymbol{E}}(\nabla_s\boldsymbol{m})^T\cdot\boldsymbol{\Omega} \\ & -2\boldsymbol{R}^T\nabla_s\boldsymbol{u}\cdot W_{\boldsymbol{EE}}[\boldsymbol{R}^T\boldsymbol{\Omega}\nabla_s\boldsymbol{m}] - 2\boldsymbol{R}^T\boldsymbol{\Omega}\nabla_s\boldsymbol{m}\cdot W_{\boldsymbol{EK}}[\boldsymbol{R}^T\nabla_s\boldsymbol{\omega}] \\ & +\boldsymbol{R}^T\boldsymbol{\Omega}\nabla_s\boldsymbol{m}\cdot W_{\boldsymbol{EE}}[\boldsymbol{R}^T\boldsymbol{\Omega}\nabla_s\boldsymbol{m}] - \boldsymbol{N}\boldsymbol{R}^T\cdot\boldsymbol{\Omega}^2.\end{aligned} \tag{47}$$

Notice that the second derivatives $W_{\boldsymbol{EE}}$ and $W_{\boldsymbol{KK}}$ possess major symmetries, and it holds $\big(W_{\boldsymbol{EK}}\big)^T = W_{\boldsymbol{KE}}$, so we have

$$\boldsymbol{A}\cdot W_{\boldsymbol{EK}}[\boldsymbol{B}] = \boldsymbol{B}\cdot W_{\boldsymbol{KE}}[\boldsymbol{A}]\,,$$

for any second order tensors $\boldsymbol{A}$, $\boldsymbol{B}$.

4 Necessary conditions for energy minimizers

In this section, we derive necessary conditions which result from the non-negativity of the second variation of the energy at equilibrium. Thus, we apply the condition (35) for variations $\boldsymbol{u}$ and $\boldsymbol{\omega}$ of a special form.

Let (θ_0^1,θ_0^2) be an arbitrary fixed interior point of $\omega\subset\mathbb{R}^2$ and $\boldsymbol{x}_0 = \boldsymbol{x}(\theta_0^1,\theta_0^2)$ be the corresponding interior point of the midsurface $\Omega\subset\mathbb{R}^3$. Let D be an arbitrary bounded open set of $\mathbb{R}^2$ and $\sigma>0$ a constant. Using a similar method as in [20, 35], we consider variations of the following form

$$\boldsymbol{u}\big(\boldsymbol{x}(\theta^1,\theta^2)\big) = \begin{cases} \sigma\,\boldsymbol{v}(\zeta^1,\zeta^2)\,, & \text{if } (\zeta^1,\zeta^2)\in D \\ \boldsymbol{0}\,, & \text{if } (\zeta^1,\zeta^2)\notin D \end{cases} \qquad \text{and}$$

$$\boldsymbol{\omega}\big(\boldsymbol{x}(\theta^1,\theta^2)\big) = \begin{cases} \sigma\,\boldsymbol{\eta}(\zeta^1,\zeta^2)\,, & \text{if } (\zeta^1,\zeta^2)\in D \\ \boldsymbol{0}\,, & \text{if } (\zeta^1,\zeta^2)\notin D \end{cases} \qquad \text{with} \quad \zeta^\alpha = \frac{\theta^\alpha-\theta_0^\alpha}{\sigma}\,, \tag{48}$$

where $\boldsymbol{v}$ and $\boldsymbol{\eta}$ are smooth vector fields compactly supported in D. We choose the constant $\sigma>0$ sufficiently small such that $(\theta_0^1,\theta_0^2)+\sigma D\subset\omega$. Then, the variations $\boldsymbol{u}$ and $\boldsymbol{\omega}$ defined by (48) vanish on the boundary $\partial\Omega$, so they are admissible fields. Moreover, we notice that the last integral vanishes in relation (46) (since $\boldsymbol{\Omega}=\boldsymbol{0}$ on the boundary), so we can write the inequality (35) in the form

$$\begin{aligned}\int_{\Omega'}\Big(&\boldsymbol{R}^T\nabla_s\boldsymbol{u}\cdot W_{\boldsymbol{EE}}[\boldsymbol{R}^T\nabla_s\boldsymbol{u}] + 2\boldsymbol{R}^T\nabla_s\boldsymbol{u}\cdot W_{\boldsymbol{EK}}[\boldsymbol{R}^T\nabla_s\boldsymbol{\omega}]\\ &+\boldsymbol{R}^T\nabla_s\boldsymbol{\omega}\cdot W_{\boldsymbol{KK}}[\boldsymbol{R}^T\nabla_s\boldsymbol{\omega}]\Big)\mathrm{d}a(\boldsymbol{x}) + \int_{\Omega'}F(\boldsymbol{m},\boldsymbol{R},\boldsymbol{u},\boldsymbol{\Omega})\,\mathrm{d}a(\boldsymbol{x})\geq 0,\end{aligned} \tag{49}$$

where Ω' is the subset of Ω given by $\Omega' = \boldsymbol{x}\big((\theta_0^1,\theta_0^2)+\sigma D\big)$. Taking into account that $\mathrm{d}a(\boldsymbol{x}) = \sqrt{a(\theta^1,\theta^2)}\,\mathrm{d}\theta^1\mathrm{d}\theta^2$, we transform the above integrals

to the domain $(\theta_0^1, \theta_0^2) + \sigma D \subset \omega$ and get

$$\int_{(\theta_0^1,\theta_0^2)+\sigma D} \Big(\boldsymbol{R}^T \nabla_s \boldsymbol{u} \cdot W_{\boldsymbol{EE}}[\boldsymbol{R}^T \nabla_s \boldsymbol{u}] + 2\boldsymbol{R}^T \nabla_s \boldsymbol{u} \cdot W_{\boldsymbol{EK}}[\boldsymbol{R}^T \nabla_s \boldsymbol{\omega}] \\ + \boldsymbol{R}^T \nabla_s \boldsymbol{\omega} \cdot W_{\boldsymbol{KK}}[\boldsymbol{R}^T \nabla_s \boldsymbol{\omega}] \Big) \sqrt{a(\theta^1, \theta^2)}\, \mathrm{d}\theta^1 \mathrm{d}\theta^2 \\ + \int_{(\theta_0^1,\theta_0^2)+\sigma D} F(\boldsymbol{m}, \boldsymbol{R}, \boldsymbol{u}, \boldsymbol{\Omega}) \sqrt{a(\theta^1, \theta^2)}\, \mathrm{d}\theta^1 \mathrm{d}\theta^2 \geq 0. \tag{50}$$

In view of (48), the gradients $\nabla_s \boldsymbol{u}$ and $\nabla_s \boldsymbol{\omega}$ are

$$\nabla_s \boldsymbol{u} = \sigma \, \frac{\partial \boldsymbol{v}}{\partial \theta^\alpha} \otimes \boldsymbol{a}^\alpha = \frac{\partial \boldsymbol{v}}{\partial \zeta^\alpha} \otimes \boldsymbol{a}^\alpha, \quad \nabla_s \boldsymbol{\omega} = \sigma \, \frac{\partial \boldsymbol{\eta}}{\partial \theta^\alpha} \otimes \boldsymbol{a}^\alpha = \frac{\partial \boldsymbol{\eta}}{\partial \zeta^\alpha} \otimes \boldsymbol{a}^\alpha \quad \text{in } D. \tag{51}$$

Since $\boldsymbol{\omega} = O(\sigma)$ by definition, we have $\boldsymbol{\Omega} = O(\sigma)$ and for the function $F(\boldsymbol{m}, \boldsymbol{R}, \boldsymbol{u}, \boldsymbol{\Omega})$ defined in (47) we get

$$F(\boldsymbol{m}, \boldsymbol{R}, \boldsymbol{u}, \boldsymbol{\Omega}) = O(\sigma), \tag{52}$$

so the last integral in (50) is of order $O(\sigma)$. If we make the substitution $\zeta^\gamma = \frac{1}{\sigma}(\theta^\gamma - \theta_0^\gamma)$ in the integrals (50), we have $\mathrm{d}\theta^1 \mathrm{d}\theta^2 = \sigma^2 \mathrm{d}\zeta^1 \mathrm{d}\zeta^2$, and the integration domain becomes D. Dividing the inequality (50) by σ^2 and using (51), we deduce

$$\int_D \Big[\big(\boldsymbol{R}^T \frac{\partial \boldsymbol{v}}{\partial \zeta^\alpha} \otimes \boldsymbol{a}^\alpha \big) \cdot W_{\boldsymbol{EE}} \big[\boldsymbol{R}^T \frac{\partial \boldsymbol{v}}{\partial \zeta^\beta} \otimes \boldsymbol{a}^\beta \big] \\ + 2 \big(\boldsymbol{R}^T \frac{\partial \boldsymbol{v}}{\partial \zeta^\alpha} \otimes \boldsymbol{a}^\alpha \big) \cdot W_{\boldsymbol{EK}} \big[\boldsymbol{R}^T \frac{\partial \boldsymbol{\eta}}{\partial \zeta^\beta} \otimes \boldsymbol{a}^\beta \big] \\ + \big(\boldsymbol{R}^T \frac{\partial \boldsymbol{\eta}}{\partial \zeta^\alpha} \otimes \boldsymbol{a}^\alpha \big) \cdot W_{\boldsymbol{KK}} \big[\boldsymbol{R}^T \frac{\partial \boldsymbol{\eta}}{\partial \zeta^\beta} \otimes \boldsymbol{a}^\beta \big] \Big] \sqrt{a(\theta_0^\gamma + \sigma \zeta^\gamma)}\, \mathrm{d}\zeta^1 \mathrm{d}\zeta^2 + O(\sigma) \geq 0. \tag{53}$$

Due to our continuity and regularity assumptions we see that the above integrand is uniformly bounded. Hence, passing to the limit $\sigma \to 0$ we obtain by the dominated convergence theorem the inequality

$$\int_D \Big[\big(\boldsymbol{R}_0^T \frac{\partial \boldsymbol{v}}{\partial \zeta^\alpha} \otimes \boldsymbol{a}_0^\alpha \big) \cdot W^0_{\boldsymbol{EE}} \big[\boldsymbol{R}_0^T \frac{\partial \boldsymbol{v}}{\partial \zeta^\beta} \otimes \boldsymbol{a}_0^\beta \big] \\ + 2 \big(\boldsymbol{R}_0^T \frac{\partial \boldsymbol{v}}{\partial \zeta^\alpha} \otimes \boldsymbol{a}_0^\alpha \big) \cdot W^0_{\boldsymbol{EK}} \big[\boldsymbol{R}_0^T \frac{\partial \boldsymbol{\eta}}{\partial \zeta^\beta} \otimes \boldsymbol{a}_0^\beta \big] \\ + \big(\boldsymbol{R}_0^T \frac{\partial \boldsymbol{\eta}}{\partial \zeta^\alpha} \otimes \boldsymbol{a}_0^\alpha \big) \cdot W^0_{\boldsymbol{KK}} \big[\boldsymbol{R}_0^T \frac{\partial \boldsymbol{\eta}}{\partial \zeta^\beta} \otimes \boldsymbol{a}_0^\beta \big] \Big] \sqrt{a(\theta_0^1, \theta_0^2)}\, \mathrm{d}\zeta^1 \mathrm{d}\zeta^2 \geq 0, \tag{54}$$

where all the fields with index 0 represent those functions evaluated at $\boldsymbol{x}_0$, such as $\boldsymbol{R}_0 = \boldsymbol{R}(\boldsymbol{x}_0)$, $\boldsymbol{a}_0^\alpha = \boldsymbol{a}^\alpha(\boldsymbol{x}_0)$, $W^0_{\boldsymbol{EE}} = \Big(W_{\boldsymbol{EE}}\Big)_{\big|\boldsymbol{x}=\boldsymbol{x}_0}$ etc. If we divide the last relation by the constant factor $\sqrt{a(\theta^1_0, \theta^2_0)}$, then we obtain the following necessary condition

$$\begin{aligned}\int_D \Big[\big(\boldsymbol{R}_0^T \frac{\partial \boldsymbol{v}}{\partial \zeta^\alpha} \otimes \boldsymbol{a}_0^\alpha\big) \cdot W^0_{\boldsymbol{EE}}\big[\boldsymbol{R}_0^T \frac{\partial \boldsymbol{v}}{\partial \zeta^\beta} \otimes \boldsymbol{a}_0^\beta\big] \\ +2\big(\boldsymbol{R}_0^T \frac{\partial \boldsymbol{v}}{\partial \zeta^\alpha} \otimes \boldsymbol{a}_0^\alpha\big) \cdot W^0_{\boldsymbol{EK}}\big[\boldsymbol{R}_0^T \frac{\partial \boldsymbol{\eta}}{\partial \zeta^\beta} \otimes \boldsymbol{a}_0^\beta\big] \\ +\big(\boldsymbol{R}_0^T \frac{\partial \boldsymbol{\eta}}{\partial \zeta^\alpha} \otimes \boldsymbol{a}_0^\alpha\big) \cdot W^0_{\boldsymbol{KK}}\big[\boldsymbol{R}_0^T \frac{\partial \boldsymbol{\eta}}{\partial \zeta^\beta} \otimes \boldsymbol{a}_0^\beta\big]\Big]\, \mathrm{d}\zeta^1 \mathrm{d}\zeta^2 \geq 0,\end{aligned} \tag{55}$$

which holds in any point $\boldsymbol{x}_0$, for any open bounded set $D \subset \mathbb{R}^2$ and for every smooth vector fields $\boldsymbol{v}(\zeta^1, \zeta^2)$, $\boldsymbol{\eta}(\zeta^1, \zeta^2)$ compactly supported in D.

Notice that the dyadic products in the inequality (55) can be regarded as surface gradients in the tangent plane $\mathcal{T}_p(\boldsymbol{x}_0)$ to the midsurface Ω in the point $\boldsymbol{x}_0$. Indeed, since the covariant basis vectors $\boldsymbol{a}_1(\boldsymbol{x}_0)$ and $\boldsymbol{a}_2(\boldsymbol{x}_0)$ span the tangent plane, we can represent the position vector $\boldsymbol{p}$ of any point in the tangent plane $\mathcal{T}_p(\boldsymbol{x}_0)$ in the form

$$\boldsymbol{p}(\zeta^1, \zeta^2) = \boldsymbol{x}_0 + \zeta^1 \boldsymbol{a}_1(\boldsymbol{x}_0) + \zeta^2 \boldsymbol{a}_2(\boldsymbol{x}_0). \tag{56}$$

We denote the image of any domain $D \subset \mathbb{R}^2$ under the map (56) by $D_p = \boldsymbol{p}(D) \subset \mathcal{T}_p(\boldsymbol{x}_0)$. For any smooth vector field $\boldsymbol{f}(\zeta^1, \zeta^2)$ compactly supported in D we define the associated vector field $\hat{\boldsymbol{f}}$ on D_p such that

$$\hat{\boldsymbol{f}}\big(\boldsymbol{p}(\zeta^1, \zeta^2)\big) = \boldsymbol{f}(\zeta^1, \zeta^2), \qquad \text{i.e.} \qquad \hat{\boldsymbol{f}} \circ \boldsymbol{p} = \boldsymbol{f}\,. \tag{57}$$

Then, since the contravariant basis vectors are $\boldsymbol{a}^\alpha(\boldsymbol{x}_0) = \boldsymbol{a}_0^\alpha$, the surface gradient ∇_t in the tangent plane $\mathcal{T}_p(\boldsymbol{x}_0)$ is given by

$$\frac{\partial \boldsymbol{f}(\zeta^1, \zeta^2)}{\partial \zeta^\alpha} \otimes \boldsymbol{a}_0^\alpha = \frac{\partial \hat{\boldsymbol{f}}\big(\boldsymbol{p}(\zeta^1, \zeta^2)\big)}{\partial \zeta^\alpha} \otimes \boldsymbol{a}_0^\alpha = \nabla_t \hat{\boldsymbol{f}}(\boldsymbol{p}). \tag{58}$$

Moreover, if $\boldsymbol{f}$ is a smooth field compactly supported in D, then $\hat{\boldsymbol{f}}$ is smooth and compactly supported in D_p. With these notations, we can make the change of variables $\boldsymbol{p} = \boldsymbol{p}(\zeta^1, \zeta^2)$ in the integral (54) and using $\mathrm{d}a(\boldsymbol{p}) = \sqrt{a(\theta^1_0, \theta^2_0)}\, \mathrm{d}\zeta^1 \mathrm{d}\zeta^2$ we obtain the following alternative form of the necessary condition (55)

$$\begin{aligned}\int_{D_p} \Big[\big(\boldsymbol{R}_0^T \nabla_t \hat{\boldsymbol{v}}\big) \cdot W^0_{\boldsymbol{EE}}\big[\boldsymbol{R}_0^T \nabla_t \hat{\boldsymbol{v}}\big] + 2\big(\boldsymbol{R}_0^T \nabla_t \hat{\boldsymbol{v}}\big) \cdot W^0_{\boldsymbol{EK}}\big[\boldsymbol{R}_0^T \nabla_t \hat{\boldsymbol{\eta}}\big] \\ +\big(\boldsymbol{R}_0^T \nabla_t \hat{\boldsymbol{\eta}}\big) \cdot W^0_{\boldsymbol{KK}}\big[\boldsymbol{R}_0^T \nabla_t \hat{\boldsymbol{\eta}}\big]\Big]\, \mathrm{d}a(\boldsymbol{p}) \geq 0,\end{aligned} \tag{59}$$

which holds in any point $\boldsymbol{x}_0 \in \Omega$, for any open bounded set D_p in the tangent plane $\mathcal{T}_p(\boldsymbol{x}_0)$ and for every smooth vector fields $\hat{\boldsymbol{v}}$, $\hat{\boldsymbol{\eta}}$ compactly supported in D_p .

Remark 2. Notice that the inequality (59) has a similar expression with the corresponding necessary condition for three-dimensional Cosserat bodies established in [20, Eq. (4.4)], whereas the Cosserat deformation tensor is replaced by the shell strain tensor $\boldsymbol{E}$, and the wryness tensor is replaced by the shell bending-curvature tensor $\boldsymbol{K}$. However, a significant difference to the three-dimensional Cosserat theory is that the necessary condition (59) must hold in the tangent plane $\mathcal{T}_p(\boldsymbol{x}_0)$.

5 Legendre-Hadamard condition for shells

In the tangent plane $\mathcal{T}_p(\boldsymbol{x}_0)$ we choose a Cartesian orthogonal coordinate system (ξ^1, ξ^2) with origin in $\boldsymbol{x}_0$ and denote the unit vectors along the coordinate axes with $\boldsymbol{s}$ and $\boldsymbol{t}$. Thus, we have $\boldsymbol{p} - \boldsymbol{x}_0 = \xi^1 \boldsymbol{s} + \xi^2 \boldsymbol{t}$, and the inequality (59) can be written in the simpler form

$$\begin{aligned}\int_{\hat{D}} \Big[\big(\boldsymbol{R}_0^T \nabla \boldsymbol{v}(\xi^1,\xi^2)\big) \cdot W^0_{\boldsymbol{EE}}\big[\boldsymbol{R}_0^T \nabla \boldsymbol{v}(\xi^1,\xi^2)\big] \\ +2\big(\boldsymbol{R}_0^T \nabla \boldsymbol{v}(\xi^1,\xi^2)\big) \cdot W^0_{\boldsymbol{EK}}\big[\boldsymbol{R}_0^T \nabla \boldsymbol{\eta}(\xi^1,\xi^2)\big] \\ +\big(\boldsymbol{R}_0^T \nabla \boldsymbol{\eta}(\xi^1,\xi^2)\big) \cdot W^0_{\boldsymbol{KK}}\big[\boldsymbol{R}_0^T \nabla \boldsymbol{\eta}(\xi^1,\xi^2)\big] \Big] \,\mathrm{d}\xi^1 \mathrm{d}\xi^2 \geq 0,\end{aligned} \tag{60}$$

where $\hat{D}$ is an arbitrary open bounded set in $\mathbb{R}^2$, while $\boldsymbol{v}$ and $\boldsymbol{\eta}$ are arbitrary smooth vector fields compactly supported in $\hat{D}$. The gradients appearing in (60) are given by

$$\nabla \boldsymbol{v}(\xi^1,\xi^2) = \frac{\partial \boldsymbol{v}}{\partial \xi^1} \otimes \boldsymbol{s} + \frac{\partial \boldsymbol{v}}{\partial \xi^2} \otimes \boldsymbol{t}\,, \qquad \nabla \boldsymbol{\eta}(\xi^1,\xi^2) = \frac{\partial \boldsymbol{\eta}}{\partial \xi^1} \otimes \boldsymbol{s} + \frac{\partial \boldsymbol{\eta}}{\partial \xi^2} \otimes \boldsymbol{t}\,. \tag{61}$$

To obtain the Legendre-Hadamard conditions for shells we follow the method presented in [35] and extend the fields $\boldsymbol{v}$ and $\boldsymbol{\eta}$ to complex-valued vector fields

$$\boldsymbol{v} = \boldsymbol{v}_1 + i\boldsymbol{v}_2 \qquad \text{and} \qquad \boldsymbol{\eta} = \boldsymbol{\eta}_1 + i\boldsymbol{\eta}_2\,. \tag{62}$$

Then, we have $\nabla \boldsymbol{v} = \nabla \boldsymbol{v}_1 + i\,\nabla \boldsymbol{v}_2$ and $\nabla \bar{\boldsymbol{\eta}} = \nabla \boldsymbol{\eta}_1 - i\,\nabla \boldsymbol{\eta}_2$, where the overline in $\bar{\boldsymbol{\eta}}$ denotes the complex conjugate. We deduce

$$\begin{aligned} \big(\boldsymbol{R}_0^T \nabla \boldsymbol{v}\big) \cdot W^0_{\boldsymbol{EE}}\big[\boldsymbol{R}_0^T \nabla \bar{\boldsymbol{v}}\big] = \; & \big(\boldsymbol{R}_0^T \nabla \boldsymbol{v}_1\big) \cdot W^0_{\boldsymbol{EE}}\big[\boldsymbol{R}_0^T \nabla \boldsymbol{v}_1\big] \\ & + \big(\boldsymbol{R}_0^T \nabla \boldsymbol{v}_2\big) \cdot W^0_{\boldsymbol{EE}}\big[\boldsymbol{R}_0^T \nabla \boldsymbol{v}_2\big], \\ \big(\boldsymbol{R}_0^T \nabla \boldsymbol{\eta}\big) \cdot W^0_{\boldsymbol{KK}}\big[\boldsymbol{R}_0^T \nabla \bar{\boldsymbol{\eta}}\big] = \; & \big(\boldsymbol{R}_0^T \nabla \boldsymbol{\eta}_1\big) \cdot W^0_{\boldsymbol{KK}}\big[\boldsymbol{R}_0^T \nabla \boldsymbol{\eta}_1\big] \\ & + \big(\boldsymbol{R}_0^T \nabla \boldsymbol{\eta}_2\big) \cdot W^0_{\boldsymbol{KK}}\big[\boldsymbol{R}_0^T \nabla \boldsymbol{\eta}_2\big], \end{aligned} \tag{63}$$

and also

$$\begin{aligned} & \big(\boldsymbol{R}_0^T \nabla \boldsymbol{v}\big) \cdot W^0_{\boldsymbol{EK}}\big[\boldsymbol{R}_0^T \nabla \bar{\boldsymbol{\eta}}\big] + \big(\boldsymbol{R}_0^T \nabla \boldsymbol{\eta}\big) \cdot W^0_{\boldsymbol{KE}}\big[\boldsymbol{R}_0^T \nabla \bar{\boldsymbol{v}}\big] \\ & \quad = 2\big(\boldsymbol{R}_0^T \nabla \boldsymbol{v}_1\big) \cdot W^0_{\boldsymbol{EK}}\big[\boldsymbol{R}_0^T \nabla \boldsymbol{\eta}_1\big] + 2\big(\boldsymbol{R}_0^T \nabla \boldsymbol{v}_2\big) \cdot W^0_{\boldsymbol{EK}}\big[\boldsymbol{R}_0^T \nabla \boldsymbol{\eta}_2\big]. \end{aligned} \tag{64}$$

Summation of relations (63) and (64), and application of the inequality (60) for the pairs of fields $\{\boldsymbol{v}_1, \boldsymbol{\eta}_1\}$ and $\{\boldsymbol{v}_2, \boldsymbol{\eta}_2\}$ yield the condition

$$\begin{aligned} \int_{\hat{D}} \Big[& \big(\boldsymbol{R}_0^T \nabla \boldsymbol{v}(\xi^1,\xi^2)\big) \cdot W^0_{\boldsymbol{EE}}\big[\boldsymbol{R}_0^T \nabla \bar{\boldsymbol{v}}(\xi^1,\xi^2)\big] \\ & + \big(\boldsymbol{R}_0^T \nabla \boldsymbol{v}(\xi^1,\xi^2)\big) \cdot W^0_{\boldsymbol{EK}}\big[\boldsymbol{R}_0^T \nabla \bar{\boldsymbol{\eta}}(\xi^1,\xi^2)\big] \\ & + \big(\boldsymbol{R}_0^T \nabla \boldsymbol{\eta}(\xi^1,\xi^2)\big) \cdot W^0_{\boldsymbol{KE}}\big[\boldsymbol{R}_0^T \nabla \bar{\boldsymbol{v}}(\xi^1,\xi^2)\big] \\ & + \big(\boldsymbol{R}_0^T \nabla \boldsymbol{\eta}(\xi^1,\xi^2)\big) \cdot W^0_{\boldsymbol{KK}}\big[\boldsymbol{R}_0^T \nabla \bar{\boldsymbol{\eta}}(\xi^1,\xi^2)\big]\Big]\, \mathrm{d}\xi^1 \mathrm{d}\xi^2 \geq 0, \end{aligned} \tag{65}$$

which holds for any complex-valued fields $\boldsymbol{v}$, $\boldsymbol{\eta}$ compactly supported in $\hat{D}$.

Now, let us choose the fields $\boldsymbol{v}$ and $\boldsymbol{\eta}$ of the following form

$$\boldsymbol{v}(\xi^1,\xi^2) = \boldsymbol{\alpha} \exp(ik\boldsymbol{\tau}\cdot\boldsymbol{\xi})\, f(\xi^1,\xi^2), \qquad \boldsymbol{\eta}(\xi^1,\xi^2) = \boldsymbol{\beta} \exp(ik\boldsymbol{\tau}\cdot\boldsymbol{\xi})\, f(\xi^1,\xi^2), \tag{66}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are arbitrary fixed vectors, $\boldsymbol{\tau} = \tau_1 \boldsymbol{s} + \tau_2 \boldsymbol{t}$ is an arbitrary fixed vector in the tangent plane, $k \neq 0$ is an arbitrary real constant, and f is an arbitrary real-valued smooth function compactly supported in $\hat{D}$. Here, we denote by $\boldsymbol{\xi}$ the vector $\boldsymbol{\xi} = \xi^1 \boldsymbol{t} + \xi^2 \boldsymbol{s}$, so we have $\boldsymbol{\tau}\cdot\boldsymbol{\xi} = \tau_1 \xi^1 + \tau_2 \xi^2$. In view of (66) we can compute the gradients

$$\begin{aligned} & \nabla \boldsymbol{v}(\xi^1,\xi^2) = \exp(ik\boldsymbol{\tau}\cdot\boldsymbol{\xi})\, \boldsymbol{\alpha} \otimes \big[\nabla f(\xi^1,\xi^2) + ikf(\xi^1,\xi^2)\boldsymbol{\tau}\big] \qquad \text{and} \\ & \nabla \boldsymbol{\eta}(\xi^1,\xi^2) = \exp(ik\boldsymbol{\tau}\cdot\boldsymbol{\xi})\, \boldsymbol{\beta} \otimes \big[\nabla f(\xi^1,\xi^2) + ikf(\xi^1,\xi^2)\boldsymbol{\tau}\big] \end{aligned} \tag{67}$$

with $\nabla f(\xi^1,\xi^2) = \frac{\partial f}{\partial \xi^1}\, \boldsymbol{s} + \frac{\partial f}{\partial \xi^2}\, \boldsymbol{t}$. If we insert this in the inequality (65) and

denote by $\boldsymbol{a} = \boldsymbol{R}_0^T \boldsymbol{\alpha}$ and $\boldsymbol{b} = \boldsymbol{R}_0^T \boldsymbol{\beta}$, then we obtain

$$\begin{aligned}\int_{\hat{D}} \Big[& \big(\boldsymbol{a}\otimes k f(\xi^1,\xi^2)\boldsymbol{\tau}\big)\cdot W^0_{\boldsymbol{EE}}\big[\boldsymbol{a}\otimes k f(\xi^1,\xi^2)\boldsymbol{\tau}\big] \\ & + \big(\boldsymbol{a}\otimes \nabla f(\xi^1,\xi^2)\big)\cdot W^0_{\boldsymbol{EE}}\big[\boldsymbol{a}\otimes \nabla f(\xi^1,\xi^2)\big] \\ & + 2\big(\boldsymbol{a}\otimes k f(\xi^1,\xi^2)\boldsymbol{\tau}\big)\cdot W^0_{\boldsymbol{EK}}\big[\boldsymbol{b}\otimes k f(\xi^1,\xi^2)\boldsymbol{\tau}\big] \\ & + 2\big(\boldsymbol{a}\otimes \nabla f(\xi^1,\xi^2)\big)\cdot W^0_{\boldsymbol{EK}}\big[\boldsymbol{b}\otimes \nabla f(\xi^1,\xi^2)\big] \\ & + \big(\boldsymbol{b}\otimes k f(\xi^1,\xi^2)\boldsymbol{\tau}\big)\cdot W^0_{\boldsymbol{KK}}\big[\boldsymbol{b}\otimes k f(\xi^1,\xi^2)\boldsymbol{\tau}\big] \\ & + \big(\boldsymbol{b}\otimes \nabla f(\xi^1,\xi^2)\big)\cdot W^0_{\boldsymbol{KK}}\big[\boldsymbol{b}\otimes \nabla f(\xi^1,\xi^2)\big]\Big]\,\mathrm{d}\xi^1\mathrm{d}\xi^2 \geq 0. \end{aligned} \tag{68}$$

Dividing the last relation by k^2, and then letting k tend to infinity ($k \to \infty$) we get

$$\begin{aligned}\Big[& (\boldsymbol{a}\otimes\boldsymbol{\tau})\cdot W^0_{\boldsymbol{EE}}[\boldsymbol{a}\otimes\boldsymbol{\tau}] + 2(\boldsymbol{a}\otimes\boldsymbol{\tau})\cdot W^0_{\boldsymbol{EK}}[\boldsymbol{b}\otimes\boldsymbol{\tau}] \\ & + (\boldsymbol{b}\otimes\boldsymbol{\tau})\cdot W^0_{\boldsymbol{KK}}[\boldsymbol{b}\otimes\boldsymbol{\tau}]\Big]\cdot \int_{\hat{D}} \big(f(\xi^1,\xi^2)\big)^2\,\mathrm{d}\xi^1\mathrm{d}\xi^2 \geq 0. \end{aligned} \tag{69}$$

Since the function f is arbitrary, we deduce that the expression in brackets from (69) is non-negative. Due to the fact that the point $\boldsymbol{x}_0$ is arbitrary, we can omit the index 0 and write the *Legendre-Hadamard condition* for shells in the final form

$$(\boldsymbol{a}\otimes\boldsymbol{\tau})\cdot W_{\boldsymbol{EE}}[\boldsymbol{a}\otimes\boldsymbol{\tau}] + 2(\boldsymbol{a}\otimes\boldsymbol{\tau})\cdot W_{\boldsymbol{EK}}[\boldsymbol{b}\otimes\boldsymbol{\tau}] + (\boldsymbol{b}\otimes\boldsymbol{\tau})\cdot W_{\boldsymbol{KK}}[\boldsymbol{b}\otimes\boldsymbol{\tau}] \geq 0. \tag{70}$$

In conclusion, we have proved that if the second variation of the potential energy is non-negative (35), then it is necessary that the Legendre-Hadamard inequality (70) holds in any point $\boldsymbol{x}$ of the midsurface Ω, for any vector $\boldsymbol{\tau}$ tangent to Ω in $\boldsymbol{x}$ (i.e., $\boldsymbol{\tau}\cdot\boldsymbol{n}(\boldsymbol{x}) = 0$), and for arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Remark 3. 1) In this inequality we can assume without loss of generality that the tangent vector $\boldsymbol{\tau}$ is unitary, i.e. $\|\boldsymbol{\tau}\| = 1$.

2) If we choose the vector $\boldsymbol{b} = \boldsymbol{0}$, then the inequality (70) reduces to

$$(\boldsymbol{a}\otimes\boldsymbol{\tau})\cdot W_{\boldsymbol{EE}}[\boldsymbol{a}\otimes\boldsymbol{\tau}] \geq 0. \tag{71}$$

Similarly, choosing $\boldsymbol{a} = \boldsymbol{0}$ in (70) yields

$$(\boldsymbol{b}\otimes\boldsymbol{\tau})\cdot W_{\boldsymbol{KK}}[\boldsymbol{b}\otimes\boldsymbol{\tau}] \geq 0. \tag{72}$$

These relations represent further necessary conditions for energy minimizers.

3) In the case when the strain energy function $W(\boldsymbol{E},\boldsymbol{K})$ is *decoupled* in $\boldsymbol{E}$ and $\boldsymbol{K}$, i.e. $W(\boldsymbol{E},\boldsymbol{K}) = W_1(\boldsymbol{E}) + W_2(\boldsymbol{K})$, then we have $W_{\boldsymbol{EK}} = 0$. Hence, the coupling term in (70) vanishes, so the Legendre-Hadamard condition is equivalent to the two decoupled inequalities (71) and (72).

4) Observe that the inequality (70) follows if the strain energy function W is convex jointly in $(\boldsymbol{E},\boldsymbol{K})$. This convexity assumption plays an important role in the proof of existence results for general 6-parameter nonlinear shells presented in [12], see also [17, 39].

5) Notice that the Legendre-Hadamard condition (70) is essentially the same as the condition for micropolar shells reported in [33, 40] (named Hadamard inequality), which has been established using another method.

6 Application to specific constitutive models

In this section we illustrate the results established above for several isotropic constitutive models of 6-parameter shells available in the literature [3, 6, 14].

Thus, we present the specific form of the strain energy density $W(\boldsymbol{E},\boldsymbol{K})$, we compute the second derivatives appearing in the Legendre-Hadamard inequality, and derive the conditions on the constitutive coefficients in each case.

6.1 Simplified isotropic 6-parameter shell model

In [3, 7] a 6-parameter model for isotropic shells made of a Cauchy continuum has been presented. Denoting by E the Young modulus, ν the Poisson ratio of the material, and h the shell thickness, then the stretching stiffness C and the bending stiffness D are given by

$$C = \frac{E\,h}{1-\nu^2} \qquad \text{and} \qquad D = \frac{E\,h^3}{12(1-\nu^2)}\,. \tag{73}$$

The strain measures $\boldsymbol{E}$ and $\boldsymbol{K}$ are decomposed into the planar and out-of-plane parts through the relations

$$\boldsymbol{E} = (\mathbb{1} + \boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{E} = \mathbb{1}\boldsymbol{E} + \boldsymbol{n}\otimes\boldsymbol{n}\boldsymbol{E} \qquad \text{and} \qquad \boldsymbol{K} = \mathbb{1}\boldsymbol{K} + \boldsymbol{n}\otimes\boldsymbol{n}\boldsymbol{K}. \tag{74}$$

With these notations, the strain energy density $W(\boldsymbol{E},\boldsymbol{K})$ in the simplified constitutive model for shells is given by [3, 7]

$$\begin{aligned} 2\,W(\boldsymbol{E},\boldsymbol{K}) = \;& C(1-\nu)\,\|\,\mathbb{1}\boldsymbol{E}\|^2 + C\nu\big(\mathrm{tr}\boldsymbol{E}\big)^2 + \alpha_s\,C(1-\nu)\,\|\,\boldsymbol{n}\boldsymbol{E}\|^2 \\ & + D(1-\nu)\,\|\,\mathbb{1}\boldsymbol{K}\|^2 + D\nu\big(\mathrm{tr}\boldsymbol{K}\big)^2 + \alpha_t\,D(1-\nu)\,\|\,\boldsymbol{n}\boldsymbol{K}\|^2, \end{aligned} \tag{75}$$

where $\alpha_s > 0$ and $\alpha_t > 0$ are two shear correction factors. The values of these shear correction factors have been identified as $\alpha_s = \frac{5}{6}$, $\alpha_t = \frac{7}{10}$ in [7, 36].

Notice that the strain energy density (75) is decoupled with respect to $\boldsymbol{E}$ and $\boldsymbol{K}$, so the mixed second derivatives vanish

$$W_{\boldsymbol{EK}} = W_{\boldsymbol{KE}} = \boldsymbol{0}. \tag{76}$$

Hence, the Legendre-Hadamard condition reduces to the two decoupled inequalities (71) and (72). To compute $W_{\boldsymbol{EE}}$ and $W_{\boldsymbol{KK}}$ we use the variational formulas

$$\begin{aligned} &\big(\|\boldsymbol{T}\|^2\big)^{\cdot} = 2\,\boldsymbol{T}\cdot\dot{\boldsymbol{T}}, \qquad \big[(\mathrm{tr}\boldsymbol{T})^2\big]^{\cdot} = 2(\mathrm{tr}\boldsymbol{T})\boldsymbol{I}\cdot\dot{\boldsymbol{T}},\\ &\big(\|\boldsymbol{nT}\|^2\big)^{\cdot} = 2(\boldsymbol{nT})\cdot(\boldsymbol{n}\dot{\boldsymbol{T}}) = 2(\boldsymbol{n}\otimes\boldsymbol{nT})\cdot\dot{\boldsymbol{T}}, \end{aligned} \tag{77}$$

and derive

$$\begin{aligned} \dot{W} = \;& \big[C(1-\nu)\,\mathbb{1}\boldsymbol{E} + C\nu\big(\mathrm{tr}\boldsymbol{E}\big)\mathbb{1} + \alpha_s\, C(1-\nu)\boldsymbol{n}\otimes\boldsymbol{nE}\big]\cdot\dot{\boldsymbol{E}}\\ &+\big[D(1-\nu)\,\mathbb{1}\boldsymbol{K} + D\nu\big(\mathrm{tr}\boldsymbol{K}\big)\mathbb{1} + \alpha_t\, D(1-\nu)\boldsymbol{n}\otimes\boldsymbol{nK}\big]\cdot\dot{\boldsymbol{K}}\,. \end{aligned} \tag{78}$$

If we compare this with the chain rule $(12)_2$, we get the first derivatives

$$\begin{aligned} W_{\boldsymbol{E}} &= C(1-\nu)\,\mathbb{1}\boldsymbol{E} + C\nu\big(\mathrm{tr}\boldsymbol{E}\big)\mathbb{1} + \alpha_s\, C(1-\nu)\boldsymbol{n}\otimes\boldsymbol{nE}\,,\\ W_{\boldsymbol{K}} &= D(1-\nu)\,\mathbb{1}\boldsymbol{K} + D\nu\big(\mathrm{tr}\boldsymbol{K}\big)\mathbb{1} + \alpha_t\, D(1-\nu)\boldsymbol{n}\otimes\boldsymbol{nK}\,, \end{aligned} \tag{79}$$

which are linear in $\boldsymbol{E}$ and $\boldsymbol{K}$, respectively. Then, the variation of relations (79) leads to

$$\begin{aligned} &W_{\boldsymbol{EE}}[\dot{\boldsymbol{E}}] = C(1-\nu)\,\mathbb{1}\dot{\boldsymbol{E}} + C\nu\big(\mathrm{tr}\dot{\boldsymbol{E}}\big)\mathbb{1} + \alpha_s\, C(1-\nu)\boldsymbol{n}\otimes\boldsymbol{n}\dot{\boldsymbol{E}} \quad \text{and}\\ &W_{\boldsymbol{KK}}[\dot{\boldsymbol{K}}] = D(1-\nu)\,\mathbb{1}\dot{\boldsymbol{K}} + D\nu\big(\mathrm{tr}\dot{\boldsymbol{K}}\big)\mathbb{1} + \alpha_t\, D(1-\nu)\boldsymbol{n}\otimes\boldsymbol{n}\dot{\boldsymbol{K}}\,. \end{aligned} \tag{80}$$

Since the variation $\dot{\boldsymbol{E}}$ is arbitrary, we deduce from $(80)_1$ that

$$W_{\boldsymbol{EE}}[\boldsymbol{T}] = C(1-\nu)\,\mathbb{1}\boldsymbol{T} + C\nu\big(\mathrm{tr}\boldsymbol{T}\big)\mathbb{1} + \alpha_s\, C(1-\nu)\boldsymbol{n}\otimes\boldsymbol{nT}, \tag{81}$$

for any tensor $\boldsymbol{T}$ of the form $\boldsymbol{T} = T_{i\alpha}\boldsymbol{a}^i\otimes\boldsymbol{a}^\alpha$. Hence, we have

$$W_{\boldsymbol{EE}}[\boldsymbol{a}\otimes\boldsymbol{\tau}] = C(1-\nu)\,\mathbb{1}\boldsymbol{a}\otimes\boldsymbol{\tau} + C\nu(\boldsymbol{a}\cdot\boldsymbol{\tau})\mathbb{1} + \alpha_s\, C(1-\nu)(\boldsymbol{a}\cdot\boldsymbol{n})\boldsymbol{n}\otimes\boldsymbol{\tau}, \tag{82}$$

and

$$\begin{aligned} (\boldsymbol{a}\otimes\boldsymbol{\tau})\cdot W_{\boldsymbol{EE}}[\boldsymbol{a}\otimes\boldsymbol{\tau}] = \;& C(1-\nu)\|\mathbb{1}\boldsymbol{a}\|^2\cdot\|\boldsymbol{\tau}\|^2 + C\nu(\boldsymbol{a}\cdot\boldsymbol{\tau})^2\\ &+\alpha_s\, C(1-\nu)(\boldsymbol{a}\cdot\boldsymbol{n})^2\cdot\|\boldsymbol{\tau}\|^2. \end{aligned} \tag{83}$$

Here we can assume without loss of generality that $\|\boldsymbol{\tau}\| = 1$ and decompose $\|\mathbb{1}\boldsymbol{a}\|^2 = (\boldsymbol{a}\cdot\boldsymbol{\tau})^2 + (\boldsymbol{a}\cdot\boldsymbol{\nu})^2$, where the vector $\boldsymbol{\nu}$ is given by $\boldsymbol{\nu} = \boldsymbol{n}\times\boldsymbol{\tau}$. Thus, we obtain

$$(\boldsymbol{a}\otimes\boldsymbol{\tau})\cdot W_{\boldsymbol{EE}}[\boldsymbol{a}\otimes\boldsymbol{\tau}] = C(\boldsymbol{a}\cdot\boldsymbol{\tau})^2 + C(1-\nu)(\boldsymbol{a}\cdot\boldsymbol{\nu})^2 + \alpha_s\, C(1-\nu)(\boldsymbol{a}\cdot\boldsymbol{n})^2. \quad (84)$$

Because the components $(\boldsymbol{a}\cdot\boldsymbol{\tau})$, $(\boldsymbol{a}\cdot\boldsymbol{\nu})$ and $(\boldsymbol{a}\cdot\boldsymbol{n})$ are arbitrary and independent, we see that the inequality (71) is satisfied if and only if

$$C \geq 0 \qquad \text{and} \qquad C(1-\nu) \geq 0. \quad (85)$$

Similarly, the inequality (72) reduces to the conditions on the coefficients

$$D \geq 0 \qquad \text{and} \qquad D(1-\nu) \geq 0. \quad (86)$$

In view of (73), the inequalities (85) and (86) are equivalent to

$$\frac{E}{1+\nu} \geq 0 \qquad \text{and} \qquad \frac{E}{1-\nu^2} \geq 0,$$

i.e.

$$\frac{E}{1+\nu} \geq 0 \qquad \text{and} \qquad \nu < 1. \quad (87)$$

Using the relations between E,ν and the Lamé constants λ,μ we can rewrite these conditions in the equivalent form

$$\mu \geq 0 \qquad \text{and} \qquad \frac{\lambda+2\mu}{\lambda+\mu} > 0. \quad (88)$$

As expected, these conditions are less restrictive than the conditions $E > 0$ and $-1 < \nu < \frac{1}{2}$ (or, equivalently, $\mu > 0$ and $3\lambda+2\mu > 0$), which characterize the coercivity of the strain energy density (75), see e.g. [12].

6.2 Isotropic 6-parameter shells with general constitutive coefficients

The local symmetry group for general 6-parameter shells has been presented in [6]. In case of isotropic shells, the following reduced form of the strain energy density $W(\boldsymbol{E},\boldsymbol{K})$ has been proposed

$$\begin{aligned} 2\,W(\boldsymbol{E},\boldsymbol{K}) = \; & \alpha_1\big[\mathrm{tr}(\mathbb{1}\boldsymbol{E})\big]^2 + \alpha_2\,\mathrm{tr}\big[(\mathbb{1}\boldsymbol{E})^2\big] + \alpha_3\,\|\mathbb{1}\boldsymbol{E}\|^2 + \alpha_4\,\|\boldsymbol{n}\boldsymbol{E}\|^2 \\ & + \beta_1\big[\mathrm{tr}(\mathbb{1}\boldsymbol{K})\big]^2 + \beta_2\,\mathrm{tr}\big[(\mathbb{1}\boldsymbol{K})^2\big] + \beta_3\,\|\mathbb{1}\boldsymbol{K}\|^2 + \beta_4\,\|\boldsymbol{n}\boldsymbol{K}\|^2, \end{aligned} \quad (89)$$

where α_k and β_k $(k = 1, 2, 3, 4)$ are constant constitutive coefficients. Since $\mathrm{tr}\big[(\mathbb{1}\boldsymbol{E})^2\big] = \|\mathrm{sym}(\mathbb{1}\boldsymbol{E})\|^2 - \|\mathrm{skw}(\mathbb{1}\boldsymbol{E})\|^2$, we can put the energy (89) in the equivalent form

$$\begin{aligned} 2\,W(\boldsymbol{E},\boldsymbol{K}) = \quad & (\alpha_2+\alpha_3)\|\mathrm{sym}(\mathbb{1}\boldsymbol{E})\|^2 + (\alpha_3-\alpha_2)\|\mathrm{skw}(\mathbb{1}\boldsymbol{E})\|^2 \\ & +\alpha_1(\mathrm{tr}\boldsymbol{E})^2 + \alpha_4\,\|\boldsymbol{n}\boldsymbol{E}\|^2 + (\beta_2+\beta_3)\|\mathrm{sym}(\mathbb{1}\boldsymbol{K})\|^2 \\ & +(\beta_3-\beta_2)\|\mathrm{skw}(\mathbb{1}\boldsymbol{K})\|^2 + \beta_1(\mathrm{tr}\boldsymbol{K})^2 + \beta_4\,\|\boldsymbol{n}\boldsymbol{K}\|^2. \end{aligned} \tag{90}$$

Using the same procedure as in Section 6.1, we compute the variation of the energy function (90) and determine the first derivatives

$$\begin{aligned} W_{\boldsymbol{E}} &= (\alpha_2+\alpha_3)\mathrm{sym}(\mathbb{1}\boldsymbol{E}) + (\alpha_3-\alpha_2)\mathrm{skw}(\mathbb{1}\boldsymbol{E}) + \alpha_1(\mathrm{tr}\boldsymbol{E})\mathbb{1} + \alpha_4\,\boldsymbol{n}\otimes\boldsymbol{n}\boldsymbol{E}, \\ W_{\boldsymbol{K}} &= (\beta_2+\beta_3)\mathrm{sym}(\mathbb{1}\boldsymbol{K}) + (\beta_3-\beta_2)\mathrm{skw}(\mathbb{1}\boldsymbol{K}) + \beta_1(\mathrm{tr}\boldsymbol{K})\mathbb{1} + \beta_4\,\boldsymbol{n}\otimes\boldsymbol{n}\boldsymbol{K}. \end{aligned} \tag{91}$$

Remark that the energy function is decoupled; hence, $W_{\boldsymbol{EK}} = W_{\boldsymbol{KE}} = \boldsymbol{0}$. Taking a further variation in relations (91) we find the second derivatives in the form

$$\begin{aligned} W_{\boldsymbol{EE}}[\boldsymbol{T}] &= (\alpha_2+\alpha_3)\mathrm{sym}(\mathbb{1}\boldsymbol{T}) + (\alpha_3-\alpha_2)\mathrm{skw}(\mathbb{1}\boldsymbol{T}) + \alpha_1(\mathrm{tr}\boldsymbol{T})\mathbb{1} + \alpha_4\boldsymbol{n}\otimes\boldsymbol{n}\boldsymbol{T}, \\ W_{\boldsymbol{KK}}[\boldsymbol{T}] &= (\beta_2+\beta_3)\mathrm{sym}(\mathbb{1}\boldsymbol{T}) + (\beta_3-\beta_2)\mathrm{skw}(\mathbb{1}\boldsymbol{T}) + \beta_1(\mathrm{tr}\boldsymbol{T})\mathbb{1} + \beta_4\boldsymbol{n}\otimes\boldsymbol{n}\boldsymbol{T}. \end{aligned} \tag{92}$$

Consequently, we have

$$W_{\boldsymbol{EE}}[\boldsymbol{a}\otimes\boldsymbol{\tau}] = \alpha_1(\boldsymbol{a}\cdot\boldsymbol{\tau})\mathbb{1} + \alpha_2\,\boldsymbol{\tau}\otimes\mathbb{1}\boldsymbol{a} + \alpha_3\,\mathbb{1}\boldsymbol{a}\otimes\boldsymbol{\tau} + \alpha_4(\boldsymbol{a}\cdot\boldsymbol{n})\boldsymbol{n}\otimes\boldsymbol{\tau}, \tag{93}$$

and

$$(\boldsymbol{a}\otimes\boldsymbol{\tau})\cdot W_{\boldsymbol{EE}}[\boldsymbol{a}\otimes\boldsymbol{\tau}] = (\alpha_1+\alpha_2)(\boldsymbol{a}\cdot\boldsymbol{\tau})^2 + \alpha_3\|\mathbb{1}\boldsymbol{a}\|^2 + \alpha_4(\boldsymbol{a}\cdot\boldsymbol{n})^2. \tag{94}$$

Introducing the vector $\boldsymbol{\nu} = \boldsymbol{n}\times\boldsymbol{\tau}$, we can decompose $\|\mathbb{1}\boldsymbol{a}\|^2 = (\boldsymbol{a}\cdot\boldsymbol{\tau})^2 + (\boldsymbol{a}\cdot\boldsymbol{\nu})^2$ and the last relation reduces to (since $\|\boldsymbol{\tau}\| = 1$)

$$(\boldsymbol{a}\otimes\boldsymbol{\tau})\cdot W_{\boldsymbol{EE}}[\boldsymbol{a}\otimes\boldsymbol{\tau}] = (\alpha_1+\alpha_2+\alpha_3)(\boldsymbol{a}\cdot\boldsymbol{\tau})^2 + \alpha_3(\boldsymbol{a}\cdot\boldsymbol{\nu})^2 + \alpha_4(\boldsymbol{a}\cdot\boldsymbol{n})^2. \tag{95}$$

Similarly, we get

$$(\boldsymbol{b}\otimes\boldsymbol{\tau})\cdot W_{\boldsymbol{KK}}[\boldsymbol{b}\otimes\boldsymbol{\tau}] = (\beta_1+\beta_2+\beta_3)(\boldsymbol{b}\cdot\boldsymbol{\tau})^2 + \beta_3(\boldsymbol{b}\cdot\boldsymbol{\nu})^2 + \beta_4(\boldsymbol{b}\cdot\boldsymbol{n})^2. \tag{96}$$

The Legendre-Hadamard conditions (71) and (72) state that the expressions (95) and (96) are non-negative for any arbitrary vectors $\boldsymbol{a}$ and

$\boldsymbol{b}$. Therefore, the coefficients of all square terms in (95) and (96) are non-negative. Thus, we obtain the following conditions on the constitutive coefficients

$$\begin{aligned} &\alpha_1+\alpha_2+\alpha_3\geq 0, \qquad \alpha_3\geq 0, \qquad \alpha_4\geq 0, \quad \text{and} \\ &\beta_1+\beta_2+\beta_3\geq 0, \qquad \beta_3\geq 0, \qquad \beta_4\geq 0. \end{aligned} \tag{97}$$

Remark 4. 1) According to the results presented in [32, 33], the strict form of the inequalities (97) coincide with the conditions for propagation of acceleration waves in micropolar shells. Thus, the strict inequalities (97) represent the strong ellipticity conditions for this constitutive model, see also [10, 40].

2) Notice that the conditions (97) express some constitutive inequalities and do not involve the deformation vector $\boldsymbol{m}$ or the microrotation tensor $\boldsymbol{R}$. This is due to the fact that the strain energy density $W(\boldsymbol{E},\boldsymbol{K})$ is quadratic, so the second derivatives with respect to its arguments have constant coefficients. However, in general the Legendre-Hadamard conditions do not impose restrictions on the constitutive function W, but rather on the configuration fields $\{\boldsymbol{m},\boldsymbol{R}\}$.

6.3 Cosserat elastic shells with coupling terms

In [14] we have derived a refined Cosserat shell model in which the expression of the strain energy density depends on the initial curvature, as well as on the material constants of three-dimensional Cosserat elasticity. This is a 6-parameter model for shells made of isotropic Cosserat materials. Let us denote by λ,μ the Lamé constants and by μ_c the Cosserat couple modulus [39, 41, 42]. We designate by $\boldsymbol{B}$ the initial curvature tensor given by

$$\begin{gathered} \boldsymbol{B}=-\nabla_s\boldsymbol{n}=-\boldsymbol{n}_{,\alpha}\otimes\boldsymbol{a}^\alpha=B_{\alpha\beta}\boldsymbol{a}^\alpha\otimes\boldsymbol{a}^\beta=B^\alpha_\beta\boldsymbol{a}_\alpha\otimes\boldsymbol{a}^\beta \\ \text{with}\quad B_{\alpha\beta}=-\boldsymbol{n}_{,\beta}\cdot\boldsymbol{a}_\alpha \end{gathered} \tag{98}$$

and let $\boldsymbol{B}^*$ be the cofactor of $\boldsymbol{B}$ in the tangent plane, which is defined by

$$\boldsymbol{B}^*=-\boldsymbol{B}+2H\mathbb{1}. \tag{99}$$

Notice that both tensors $\boldsymbol{B}$ and $\boldsymbol{B}^*$ are symmetric, and they satisfy $\boldsymbol{B}\boldsymbol{B}^*=K\mathbb{1}$. Here we denote by $H=\frac{1}{2}\mathrm{tr}\,\boldsymbol{B}=\frac{1}{2}B^\alpha_\alpha$ the mean curvature and by $K=\det\big(B^\alpha_\beta\big)_{2\times2}$ the Gauß curvature of the reference midsurface Ω. We also introduce the skew tensor $\boldsymbol{C}$ having the axial vector $-\boldsymbol{n}$, i.e.

$$\boldsymbol{C}=-\boldsymbol{n}\times\boldsymbol{I}=-\boldsymbol{n}\times\mathbb{1}, \tag{100}$$

which can also be represented as

$$\boldsymbol{C} = \frac{1}{\sqrt{a}}\, \epsilon_{\alpha\beta}\, \boldsymbol{a}_\alpha \otimes \boldsymbol{a}_\beta = \sqrt{a}\, \epsilon_{\alpha\beta}\, \boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta, \tag{101}$$

where $\epsilon_{\alpha\beta}$ is the two-dimensional alternator ($\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$). With these notations, the areal strain energy density of the Cosserat shell is given by [14]

$$\begin{aligned} W(\boldsymbol{E},\boldsymbol{K}) = \Big(h - K\,\frac{h^3}{12}\Big)\big[W^{\text{Coss}}(\boldsymbol{E}) + W^{\text{curv}}(\boldsymbol{K})\big] \\ + \frac{h^3}{12}\,\big[W^{\text{Coss}}(\boldsymbol{E}\boldsymbol{B}+\boldsymbol{C}\boldsymbol{K}) - 2W^{\text{Coss}}(\boldsymbol{E}\,,\,\boldsymbol{C}\boldsymbol{K}\boldsymbol{B}^*) + W^{\text{curv}}(\boldsymbol{K}\boldsymbol{B})\big], \end{aligned} \tag{102}$$

where the quadratic form $W^{\text{Coss}}(\cdot)$ and the bilinear form $W^{\text{Coss}}(\cdot\,,\cdot)$ are defined for any tensors of the form $\boldsymbol{X} = X_{i\alpha}\boldsymbol{a}^i \otimes \boldsymbol{a}^\alpha$, $\boldsymbol{Y} = Y_{i\alpha}\boldsymbol{a}^i \otimes \boldsymbol{a}^\alpha$ by

$$\begin{aligned} W^{\text{Coss}}(\boldsymbol{X}) = \;& \mu\|\text{sym}(\mathbb{1}\boldsymbol{X})\|^2 + \mu_c\|\text{skw}(\mathbb{1}\boldsymbol{X})\|^2 + \frac{\lambda\,\mu}{\lambda+2\mu}\big(\text{tr}\boldsymbol{X}\big)^2 \\ & + \frac{2\mu\mu_c}{\mu+\mu_c}\,\|\boldsymbol{n}\boldsymbol{X}\|^2, \\ W^{\text{Coss}}(\boldsymbol{X},\boldsymbol{Y}) = \;& \mu\,\text{sym}(\mathbb{1}\boldsymbol{X})\cdot\text{sym}(\mathbb{1}\boldsymbol{Y}) + \mu_c\,\text{skw}(\mathbb{1}\boldsymbol{X})\cdot\text{skw}(\mathbb{1}\boldsymbol{Y}) \\ & + \frac{\lambda\,\mu}{\lambda+2\mu}\,\big(\text{tr}\boldsymbol{X}\big)\big(\text{tr}\boldsymbol{Y}\big) + \frac{2\mu\,\mu_c}{\mu+\mu_c}\,\big(\boldsymbol{n}\boldsymbol{X}\big)\cdot\big(\boldsymbol{n}\boldsymbol{Y}\big), \end{aligned} \tag{103}$$

with $W^{\text{Coss}}(\boldsymbol{X}) = W^{\text{Coss}}(\boldsymbol{X},\boldsymbol{X})$. Also, the quadratic form $W^{\text{curv}}(\cdot)$ appearing in (102) is defined by

$$\begin{aligned} W^{\text{curv}}(\boldsymbol{X}) = a_1\|\text{sym}(\mathbb{1}\boldsymbol{X})\|^2 + a_2\|\text{skw}(\mathbb{1}\boldsymbol{X})\|^2 + \big(a_3 - \frac{a_1}{3}\big)(\text{tr}\boldsymbol{X})^2 \\ + \frac{a_1+a_2}{2}\,\|\boldsymbol{n}\boldsymbol{X}\|^2, \end{aligned} \tag{104}$$

where the coefficients a_1, a_2, a_3 are constitutive constants. Thus, we see that the strain energy function (102) is quadratic, but the model is geometrically nonlinear.

Let us derive the specific form of the Legendre-Hadamard condition (70) for this shell model. To this aim, we compute the derivatives $W_{\boldsymbol{E}}$ and $W_{\boldsymbol{K}}$, as well as the second derivatives $W_{\boldsymbol{E}\boldsymbol{E}}$, $W_{\boldsymbol{E}\boldsymbol{K}}$, $W_{\boldsymbol{K}\boldsymbol{E}}$ and $W_{\boldsymbol{K}\boldsymbol{K}}$. For the quadratic forms $W^{\text{Coss}}(\boldsymbol{X})$ and $W^{\text{curv}}(\boldsymbol{X})$ defined above we determine the first derivatives as in the previous sections and obtain

$$W^{\text{Coss}}_{\boldsymbol{X}}(\boldsymbol{T}) = 2\mu\text{sym}(\mathbb{1}\boldsymbol{T}) + 2\mu_c\text{skw}(\mathbb{1}\boldsymbol{T}) + \frac{2\lambda\,\mu}{\lambda+2\mu}\big(\text{tr}\boldsymbol{T}\big)\mathbb{1} + \frac{4\mu\,\mu_c}{\mu+\mu_c}(\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{T}, \tag{105}$$

for any $\boldsymbol{T} = T_{i\alpha}\boldsymbol{a}^i \otimes \boldsymbol{a}^\alpha$, and

$$W^{\text{curv}}_{\boldsymbol{X}}(\boldsymbol{T}) = 2a_1\text{sym}(\mathbb{1}\boldsymbol{T}) + 2a_2\text{skw}(\mathbb{1}\boldsymbol{T}) + 2\big(a_3 - \frac{a_1}{3}\big)(\text{tr}\boldsymbol{T})\mathbb{1} + (a_1+a_2)(\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{T}. \tag{106}$$

For the bilinear form $W^{\text{Coss}}(\boldsymbol{X}\,,\boldsymbol{Y})$ we write the first variation as

$$\big[W^{\text{Coss}}(\boldsymbol{T},\boldsymbol{S})\big]^{\boldsymbol{\cdot}} = W^{\text{Coss}}_{\boldsymbol{X}}(\boldsymbol{T},\boldsymbol{S})\cdot\dot{\boldsymbol{T}} + W^{\text{Coss}}_{\boldsymbol{Y}}(\boldsymbol{T},\boldsymbol{S})\cdot\dot{\boldsymbol{S}}, \tag{107}$$

where $\boldsymbol{T} = T_{i\alpha}\boldsymbol{a}^i \otimes \boldsymbol{a}^\alpha$ and $\boldsymbol{S} = S_{i\alpha}\boldsymbol{a}^i \otimes \boldsymbol{a}^\alpha$ are second order tensors. At the same time, using the expression $(103)_2$ we can compute this variation directly in the form

$$\begin{aligned}\big[W^{\text{Coss}}(\boldsymbol{T},\boldsymbol{S})\big]^{\boldsymbol{\cdot}} &= \mu\,\text{sym}(\mathbb{1}\dot{\boldsymbol{T}})\cdot\text{sym}(\mathbb{1}\boldsymbol{S}) + \mu\,\text{sym}(\mathbb{1}\boldsymbol{T})\cdot\text{sym}(\mathbb{1}\dot{\boldsymbol{S}})\\ &+\mu_c\,\text{skw}(\mathbb{1}\dot{\boldsymbol{T}})\cdot\text{skw}(\mathbb{1}\boldsymbol{S}) + \mu_c\,\text{skw}(\mathbb{1}\boldsymbol{T})\cdot\text{skw}(\mathbb{1}\dot{\boldsymbol{S}})\\ &+\frac{\lambda\,\mu}{\lambda+2\mu}\big[\big(\text{tr}\dot{\boldsymbol{T}}\big)\big(\text{tr}\boldsymbol{S}\big)+\big(\text{tr}\boldsymbol{T}\big)\big(\text{tr}\dot{\boldsymbol{S}}\big)\big] + \frac{2\mu\,\mu_c}{\mu+\mu_c}\big[\big(\boldsymbol{n}\dot{\boldsymbol{T}}\big)\cdot\big(\boldsymbol{n}\boldsymbol{S}\big) + \big(\boldsymbol{n}\boldsymbol{T}\big)\cdot\big(\boldsymbol{n}\dot{\boldsymbol{S}}\big)\big],\end{aligned}$$

or equivalently,

$$\begin{aligned}\big[W^{\text{Coss}}(\boldsymbol{T},\boldsymbol{S})\big]^{\boldsymbol{\cdot}} &= \big[\mu\,\text{sym}(\mathbb{1}\boldsymbol{S}) + \mu_c\text{skw}(\mathbb{1}\boldsymbol{S}) + \frac{\lambda\,\mu}{\lambda+2\mu}\big(\text{tr}\,\boldsymbol{S}\big)\mathbb{1}\\ &+\frac{2\mu\,\mu_c}{\mu+\mu_c}(\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{S}\big]\cdot\dot{\boldsymbol{T}} + \big[\mu\,\text{sym}(\mathbb{1}\boldsymbol{T}) + \mu_c\text{skw}(\mathbb{1}\boldsymbol{T})\\ &+\frac{\lambda\,\mu}{\lambda+2\mu}\big(\text{tr}\boldsymbol{T}\big)\mathbb{1} + \frac{2\mu\,\mu_c}{\mu+\mu_c}(\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{T}\big]\cdot\dot{\boldsymbol{S}}.\end{aligned} \tag{108}$$

Comparing (107) and (108) we obtain the derivatives

$$\begin{aligned}W^{\text{Coss}}_{\boldsymbol{X}}(\boldsymbol{T},\boldsymbol{S}) &= \mu\,\text{sym}(\mathbb{1}\boldsymbol{S}) + \mu_c\text{skw}(\mathbb{1}\boldsymbol{S}) + \tfrac{\lambda\,\mu}{\lambda+2\mu}\big(\text{tr}\,\boldsymbol{S}\big)\mathbb{1} + \tfrac{2\mu\,\mu_c}{\mu+\mu_c}(\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{S},\\ W^{\text{Coss}}_{\boldsymbol{Y}}(\boldsymbol{T},\boldsymbol{S}) &= \mu\,\text{sym}(\mathbb{1}\boldsymbol{T}) + \mu_c\text{skw}(\mathbb{1}\boldsymbol{T}) + \tfrac{\lambda\,\mu}{\lambda+2\mu}\big(\text{tr}\,\boldsymbol{T}\big)\mathbb{1} + \tfrac{2\mu\,\mu_c}{\mu+\mu_c}(\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{T}.\end{aligned} \tag{109}$$

We are now in a position to determine the variation of the shell strain energy density

$$\big[W(\boldsymbol{E},\boldsymbol{K})\big]^{\boldsymbol{\cdot}} = W_{\boldsymbol{E}}(\boldsymbol{E},\boldsymbol{K})\cdot\dot{\boldsymbol{E}} + W_{\boldsymbol{K}}(\boldsymbol{E},\boldsymbol{K})\cdot\dot{\boldsymbol{K}}. \tag{110}$$

Indeed, using the formulas (105), (106) and (109) we differentiate the relation (102) and get

$$\begin{aligned}\big[W(\boldsymbol{E},\boldsymbol{K})\big]^{\boldsymbol{\cdot}} &= \Big(h - K\,\frac{h^3}{12}\Big)\big[W^{\text{Coss}}_{\boldsymbol{X}}(\boldsymbol{E})\cdot\dot{\boldsymbol{E}} + W^{\text{curv}}_{\boldsymbol{X}}(\boldsymbol{K})\cdot\dot{\boldsymbol{K}}\big]\\ &+\frac{h^3}{12}\,\big[W^{\text{Coss}}_{\boldsymbol{X}}(\boldsymbol{E}\boldsymbol{B}+\boldsymbol{C}\boldsymbol{K})\cdot(\dot{\boldsymbol{E}}\boldsymbol{B}+\boldsymbol{C}\dot{\boldsymbol{K}}) - 2W^{\text{Coss}}_{\boldsymbol{X}}(\boldsymbol{E}\,,\boldsymbol{C}\boldsymbol{K}\boldsymbol{B}^*)\cdot\dot{\boldsymbol{E}}\\ &-2W^{\text{Coss}}_{\boldsymbol{Y}}(\boldsymbol{E}\,,\boldsymbol{C}\boldsymbol{K}\boldsymbol{B}^*)\cdot(\boldsymbol{C}\dot{\boldsymbol{K}}\boldsymbol{B}^*) + W^{\text{curv}}_{\boldsymbol{X}}(\boldsymbol{K}\boldsymbol{B})\cdot(\dot{\boldsymbol{K}}\boldsymbol{B})\big],\end{aligned}$$

and rearranging the terms we deduce

$$\begin{aligned}\big[W(\boldsymbol{E},\boldsymbol{K})\big]^{\cdot} = \Big[\Big(h - K\,\frac{h^3}{12}\Big)W_{\boldsymbol{X}}^{\text{Coss}}(\boldsymbol{E}) + \frac{h^3}{12}\,W_{\boldsymbol{X}}^{\text{Coss}}(\boldsymbol{EB}+\boldsymbol{CK})\boldsymbol{B}\\ -\frac{2h^3}{12}W_{\boldsymbol{X}}^{\text{Coss}}(\boldsymbol{E},\boldsymbol{CKB}^*)\Big]\cdot\dot{\boldsymbol{E}} + \Big[\Big(h-K\frac{h^3}{12}\Big)W_{\boldsymbol{X}}^{\text{curv}}(\boldsymbol{K})+\frac{h^3}{12}W_{\boldsymbol{X}}^{\text{curv}}(\boldsymbol{KB})\boldsymbol{B}\\ -\frac{h^3}{12}\,\boldsymbol{C}W_{\boldsymbol{X}}^{\text{Coss}}(\boldsymbol{EB}+\boldsymbol{CK}) + \frac{2h^3}{12}\,\boldsymbol{C}W_{\boldsymbol{Y}}^{\text{Coss}}(\boldsymbol{E}\,,\,\boldsymbol{CKB}^*)\boldsymbol{B}^*\Big]\cdot\dot{\boldsymbol{K}}.\end{aligned} \tag{111}$$

Hence, from (110) and (111) we find

$$\begin{aligned}W_{\boldsymbol{E}}(\boldsymbol{E},\boldsymbol{K}) = \;& \Big(h - K\,\frac{h^3}{12}\Big)W_{\boldsymbol{X}}^{\text{Coss}}(\boldsymbol{E}) + \frac{h^3}{12}\,W_{\boldsymbol{X}}^{\text{Coss}}(\boldsymbol{EB}+\boldsymbol{CK})\boldsymbol{B}\\ & -\frac{2h^3}{12}\,W_{\boldsymbol{X}}^{\text{Coss}}(\boldsymbol{E}\,,\,\boldsymbol{CKB}^*)\end{aligned} \tag{112}$$

and

$$\begin{aligned}W_{\boldsymbol{K}}(\boldsymbol{E},\boldsymbol{K}) = \Big(h - K\,\frac{h^3}{12}\Big)W_{\boldsymbol{X}}^{\text{curv}}(\boldsymbol{K}) + \frac{h^3}{12}\,W_{\boldsymbol{X}}^{\text{curv}}(\boldsymbol{KB})\boldsymbol{B}\\ -\frac{h^3}{12}\,\boldsymbol{C}W_{\boldsymbol{X}}^{\text{Coss}}(\boldsymbol{EB}+\boldsymbol{CK}) + \frac{2h^3}{12}\,\boldsymbol{C}W_{\boldsymbol{Y}}^{\text{Coss}}(\boldsymbol{E}\,,\,\boldsymbol{CKB}^*)\boldsymbol{B}^*.\end{aligned} \tag{113}$$

A further variation in the relations (112) and (113) leads to

$$\begin{aligned}\big[W_{\boldsymbol{E}}(\boldsymbol{E},\boldsymbol{K})\big]^{\cdot} &= W_{\boldsymbol{EE}}\big[\dot{\boldsymbol{E}}\big] + W_{\boldsymbol{EK}}\big[\dot{\boldsymbol{K}}\big] \qquad \text{and}\\ \big[W_{\boldsymbol{K}}(\boldsymbol{E},\boldsymbol{K})\big]^{\cdot} &= W_{\boldsymbol{KE}}\big[\dot{\boldsymbol{E}}\big] + W_{\boldsymbol{KK}}\big[\dot{\boldsymbol{K}}\big].\end{aligned} \tag{114}$$

In view of the fact that $W_{\boldsymbol{X}}^{\text{Coss}}(\cdot)$ and $W_{\boldsymbol{X}}^{\text{curv}}(\cdot)$ are linear forms of their arguments, we can compute the variational derivative of relation (112) in the form

$$\begin{aligned}\big[W_{\boldsymbol{E}}(\boldsymbol{E},\boldsymbol{K})\big]^{\cdot} = \;& \Big(h - K\,\frac{h^3}{12}\Big)W_{\boldsymbol{X}}^{\text{Coss}}(\dot{\boldsymbol{E}}) + \frac{h^3}{12}\,W_{\boldsymbol{X}}^{\text{Coss}}(\dot{\boldsymbol{E}}\boldsymbol{B}+\boldsymbol{C}\dot{\boldsymbol{K}})\boldsymbol{B}\\ & -\frac{2h^3}{12}\,W_{\boldsymbol{X}}^{\text{Coss}}(\boldsymbol{E}\,,\,\boldsymbol{C}\dot{\boldsymbol{K}}\boldsymbol{B}^*),\end{aligned}$$

which means

$$\begin{aligned}\big[W_{\boldsymbol{E}}(\boldsymbol{E},\boldsymbol{K})\big]^{\cdot} = \;& \Big[\Big(h - K\,\tfrac{h^3}{12}\Big)W_{\boldsymbol{X}}^{\text{Coss}}(\dot{\boldsymbol{E}}) + \tfrac{h^3}{12}\,W_{\boldsymbol{X}}^{\text{Coss}}(\dot{\boldsymbol{E}}\boldsymbol{B})\boldsymbol{B}\Big]\\ & +\frac{h^3}{12}\,\big[W_{\boldsymbol{X}}^{\text{Coss}}(\boldsymbol{C}\dot{\boldsymbol{K}})\boldsymbol{B} - W_{\boldsymbol{X}}^{\text{Coss}}(\boldsymbol{C}\dot{\boldsymbol{K}}\boldsymbol{B}^*)\big].\end{aligned} \tag{115}$$

Similarly, from the relation (113) we get

$$\begin{aligned}\big[W_{\boldsymbol{K}}(\boldsymbol{E},\boldsymbol{K})\big]^{\cdot} = \frac{h^3}{12}\,\big[\boldsymbol{C}W^{\text{Coss}}_{\boldsymbol{X}}(\dot{\boldsymbol{E}})\boldsymbol{B}^* - \boldsymbol{C}W^{\text{Coss}}_{\boldsymbol{X}}(\dot{\boldsymbol{E}}\boldsymbol{B})\big]\\ +\Big[\Big(h-K\frac{h^3}{12}\Big)W^{\text{curv}}_{\boldsymbol{X}}(\dot{\boldsymbol{K}}) + \frac{h^3}{12}W^{\text{curv}}_{\boldsymbol{X}}(\dot{\boldsymbol{K}}\boldsymbol{B})\boldsymbol{B} - \frac{h^3}{12}\boldsymbol{C}W^{\text{Coss}}_{\boldsymbol{X}}(\boldsymbol{C}\dot{\boldsymbol{K}})\Big].\end{aligned} \tag{116}$$

Since the variation $\dot{\boldsymbol{E}}$ and $\dot{\boldsymbol{K}}$ are arbitrary, we deduce from (114)–(116) that the second derivatives are given by

$$\begin{aligned}W_{\boldsymbol{EE}}[\boldsymbol{T}] &= \Big(h-K\,\frac{h^3}{12}\Big)W^{\text{Coss}}_{\boldsymbol{X}}(\boldsymbol{T}) + \frac{h^3}{12}\,W^{\text{Coss}}_{\boldsymbol{X}}(\boldsymbol{TB})\boldsymbol{B},\\ W_{\boldsymbol{EK}}[\boldsymbol{T}] &= \frac{h^3}{12}\,\big[W^{\text{Coss}}_{\boldsymbol{X}}(\boldsymbol{CT})\boldsymbol{B} - W^{\text{Coss}}_{\boldsymbol{X}}(\boldsymbol{CTB}^*)\big],\\ W_{\boldsymbol{KE}}[\boldsymbol{T}] &= \frac{h^3}{12}\,\big[\boldsymbol{C}W^{\text{Coss}}_{\boldsymbol{X}}(\boldsymbol{T})\boldsymbol{B}^* - \boldsymbol{C}W^{\text{Coss}}_{\boldsymbol{X}}(\boldsymbol{TB})\big],\\ W_{\boldsymbol{KK}}[\boldsymbol{T}] &= \Big(h-K\frac{h^3}{12}\Big)W^{\text{curv}}_{\boldsymbol{X}}(\boldsymbol{T}) + \frac{h^3}{12}W^{\text{curv}}_{\boldsymbol{X}}(\boldsymbol{TB})\boldsymbol{B} - \frac{h^3}{12}\boldsymbol{C}W^{\text{Coss}}_{\boldsymbol{X}}(\boldsymbol{CT}),\end{aligned} \tag{117}$$

for any tensor $\boldsymbol{T} = T_{i\alpha}\boldsymbol{a}^i\otimes\boldsymbol{a}^\alpha$. Using the expression (105) in the derivative $(117)_1$ we find

$$\begin{aligned}W_{\boldsymbol{EE}}[\boldsymbol{T}] = &\ \Big[(\mu+\mu_c)\mathbb{1} + \frac{4\mu\,\mu_c}{\mu+\mu_c}\,\boldsymbol{n}\otimes\boldsymbol{n}\Big]\Big[\Big(h-K\,\frac{h^3}{12}\Big)\boldsymbol{T} + \frac{h^3}{12}\,\boldsymbol{TB}^2\Big]\\ &+(\mu-\mu_c)\Big[\Big(h-K\,\frac{h^3}{12}\Big)\boldsymbol{T}^t\mathbb{1} + \frac{h^3}{12}\,\boldsymbol{BT}^t\boldsymbol{B}\Big]\\ &+\frac{2\lambda\,\mu}{\lambda+2\mu}\Big[\Big(h-K\,\frac{h^3}{12}\Big)(\operatorname{tr}\boldsymbol{T})\mathbb{1} + \frac{h^3}{12}\operatorname{tr}(\boldsymbol{TB})\boldsymbol{B}\Big].\end{aligned} \tag{118}$$

Substituting $\boldsymbol{T} = \boldsymbol{a}\otimes\boldsymbol{\tau}$ in the last equation, we arrive at

$$\begin{aligned}W_{\boldsymbol{EE}}[\boldsymbol{a}\otimes\boldsymbol{\tau}] = &\ \Big[(\mu+\mu_c)\mathbb{1} + \tfrac{4\mu\,\mu_c}{\mu+\mu_c}\boldsymbol{n}\otimes\boldsymbol{n}\Big]\Big[\Big(h-K\,\tfrac{h^3}{12}\Big)\boldsymbol{a}\otimes\boldsymbol{\tau} + \tfrac{h^3}{12}\boldsymbol{a}\otimes\boldsymbol{\tau}\boldsymbol{B}^2\Big]\\ &+(\mu-\mu_c)\Big[\Big(h-K\,\tfrac{h^3}{12}\Big)\boldsymbol{\tau}\otimes\mathbb{1}\boldsymbol{a} + \tfrac{h^3}{12}\boldsymbol{B\tau}\otimes\boldsymbol{Ba}\Big]\\ &+\tfrac{2\lambda\,\mu}{\lambda+2\mu}\Big[\Big(h-K\tfrac{h^3}{12}\Big)(\boldsymbol{a}\cdot\boldsymbol{\tau})\mathbb{1} + \tfrac{h^3}{12}(\boldsymbol{a}\cdot\boldsymbol{B\tau})\boldsymbol{B}\Big],\end{aligned} \tag{119}$$

since the vector $\boldsymbol{\tau}$ is orthogonal to $\boldsymbol{n}$. Analogously, we derive the relations

$$W_{\boldsymbol{KE}}[\boldsymbol{a}\otimes\boldsymbol{\tau}] = \frac{h^3}{12}\Big\{(\mu{+}\mu_c)\boldsymbol{C}(\boldsymbol{a}\otimes\boldsymbol{\tau})(\boldsymbol{B}^*{-}\boldsymbol{B}) + (\mu{-}\mu_c)\big[\boldsymbol{C}\boldsymbol{B}^*(\boldsymbol{\tau}\otimes\boldsymbol{a})\mathbb{1} \\ -\boldsymbol{C}(\boldsymbol{\tau}\otimes\boldsymbol{a})\boldsymbol{B}\big] + \frac{2\lambda\,\mu}{\lambda+2\mu}\big[(\boldsymbol{a}\cdot\boldsymbol{\tau})\boldsymbol{C}\boldsymbol{B}^* - (\boldsymbol{a}\cdot\boldsymbol{B}\boldsymbol{\tau})\boldsymbol{C}\big]\Big\}, \tag{120}$$

and

$$W_{\boldsymbol{KK}}[\boldsymbol{b}\otimes\boldsymbol{\tau}] = (a_1+a_2)\Big[\Big(h-K\,\tfrac{h^3}{12}\Big)\boldsymbol{b}\otimes\boldsymbol{\tau} + \tfrac{h^3}{12}\,\boldsymbol{b}\otimes\boldsymbol{\tau}\boldsymbol{B}^2\Big] \\ +(a_1-a_2)\Big[\Big(h-K\,\frac{h^3}{12}\Big)\boldsymbol{\tau}\otimes\mathbb{1}\boldsymbol{b} + \frac{h^3}{12}\boldsymbol{B}\boldsymbol{\tau}\otimes\boldsymbol{B}\boldsymbol{b}\Big] \\ +2\big(a_3-\ \frac{a_1}{3}\big)\Big[\Big(h{-}K\frac{h^3}{12}\Big)(\boldsymbol{b}\cdot\boldsymbol{\tau})\mathbb{1} + \frac{h^3}{12}(\boldsymbol{b}\cdot\boldsymbol{B}\boldsymbol{\tau})\boldsymbol{B}\Big] \\ +\frac{h^3}{12}\Big[(\mu+\mu_c)\mathbb{1}\boldsymbol{b}\otimes\boldsymbol{\tau} + (\mu-\mu_c)\boldsymbol{C}\boldsymbol{\tau}\otimes\boldsymbol{b}\boldsymbol{C} - \frac{2\lambda\,\mu}{\lambda+2\mu}(\boldsymbol{C}\boldsymbol{b}\cdot\boldsymbol{\tau})\boldsymbol{C}\Big]. \tag{121}$$

Further, we can compute the inner products appearing in the Legendre-Hadamard condition (70) and obtain

$$(\boldsymbol{a}\otimes\boldsymbol{\tau})\cdot W_{\boldsymbol{EE}}[\boldsymbol{a}\otimes\boldsymbol{\tau}] = \Big[(\mu{+}\mu_c)\|\mathbb{1}\boldsymbol{a}\|^2 + \frac{4\mu\,\mu_c}{\mu{+}\mu_c}(\boldsymbol{a}\cdot\boldsymbol{n})^2\Big]\Big[\Big(h{-}K\frac{h^3}{12}\Big)\|\boldsymbol{\tau}\|^2 \\ +\frac{h^3}{12}\,\|\boldsymbol{B}\boldsymbol{\tau}\|^2\Big] + \Big(\frac{\mu(3\lambda{+}2\mu)}{\lambda+2\mu} - \mu_c\Big)\Big[\Big(h-K\,\frac{h^3}{12}\Big)(\boldsymbol{a}\cdot\boldsymbol{\tau})^2 + \frac{h^3}{12}(\boldsymbol{a}\cdot\boldsymbol{B}\boldsymbol{\tau})^2\Big], \tag{122}$$

and

$$(\boldsymbol{a}\otimes\boldsymbol{\tau})\cdot W_{\boldsymbol{EK}}[\boldsymbol{b}\otimes\boldsymbol{\tau}] = (\boldsymbol{b}\otimes\boldsymbol{\tau})\cdot W_{\boldsymbol{KE}}[\boldsymbol{a}\otimes\boldsymbol{\tau}] \\ = \frac{h^3}{12}\Big\{(\mu+\mu_c)(\boldsymbol{a}\cdot\boldsymbol{C}\boldsymbol{b})\big[\boldsymbol{\tau}\cdot(\boldsymbol{B}-\boldsymbol{B}^*)\boldsymbol{\tau}\big] \\ +\Big(\frac{\mu(3\lambda+2\mu)}{\lambda+2\mu} - \mu_c\Big)\big[(\boldsymbol{a}\cdot\boldsymbol{\tau})\big(\boldsymbol{\tau}\cdot\boldsymbol{b}\boldsymbol{C}\boldsymbol{B}^*\big) - (\boldsymbol{a}\cdot\boldsymbol{B}\boldsymbol{\tau})(\boldsymbol{\tau}\cdot\boldsymbol{b}\boldsymbol{C})\big]\Big\}, \tag{123}$$

and

$$(\boldsymbol{b}{\otimes}\boldsymbol{\tau})\cdot W_{\boldsymbol{KK}}[\boldsymbol{b}{\otimes}\boldsymbol{\tau}] = \tfrac{h^3}{12}\Big[(\mu{+}\mu_c)\|\mathbb{1}\boldsymbol{b}\|^2\cdot\|\boldsymbol{\tau}\|^2 + \Big(\tfrac{\mu(3\lambda+2\mu)}{\lambda+2\mu}{-}\mu_c\Big)(\boldsymbol{\tau}\cdot\boldsymbol{b}\boldsymbol{C})^2\Big] \\ +(a_1+a_2)\|\boldsymbol{b}\|^2\Big[\Big(h-K\,\frac{h^3}{12}\Big)\|\boldsymbol{\tau}\|^2 + \frac{h^3}{12}\,\|\boldsymbol{B}\boldsymbol{\tau}\|^2\Big] \\ +\Big(\frac{a_1}{3} - a_2 + 2a_3\Big)\Big[\Big(h-K\frac{h^3}{12}\Big)(\boldsymbol{b}\cdot\boldsymbol{\tau})^2 + \frac{h^3}{12}(\boldsymbol{b}\cdot\boldsymbol{B}\boldsymbol{\tau})^2\Big]. \tag{124}$$

To simplify these expressions, we introduce the auxiliary notations α, β, γ, δ, p and q for the coefficients appearing in (122)-(124), more precisely let

$$\begin{aligned} \alpha = \mu + \mu_c\,, \qquad \beta = \frac{\mu(3\lambda+2\mu)}{\lambda+2\mu} - \mu_c\,, \qquad \gamma = \frac{4\mu\,\mu_c}{\mu+\mu_c}\,, \\ \delta = \frac{12}{h^2} - K, \qquad p = a_1 + a_2\,, \qquad q = \frac{a_1}{3} - a_2 + 2a_3\,. \end{aligned} \tag{125}$$

Notice that δ is a positive geometric parameter satisfying the relation

$$\delta = \frac{12}{h^2}\Big[1 - \frac{h^2}{12}(\kappa_1\kappa_2)\Big] = \frac{12}{h^2}\Big[1 - \frac{1}{12}(\kappa_1 h)(\kappa_2 h)\Big] > 0, \tag{126}$$

where $\kappa_1\,,\kappa_2$ are the principal curvatures of the shell, with $\kappa_1\kappa_2 = K$ and $|\kappa_\alpha h| \ll 1$.

Substituting the equations (122)-(125) in the inequality (70), we arrive at the following form of the Legendre-Hadamard condition for the considered Cosserat shell model

$$\begin{aligned} \big[\alpha\|\mathbb{1}\boldsymbol{a}\|^2 + \gamma(\boldsymbol{a}\cdot\boldsymbol{n})^2\big]\big(\delta\|\boldsymbol{\tau}\|^2 + \|\boldsymbol{B\tau}\|^2\big) + \beta\big[\delta(\boldsymbol{a}\cdot\boldsymbol{\tau})^2 + (\boldsymbol{a}\cdot\boldsymbol{B\tau})^2\big] \\ +2\alpha(\boldsymbol{a}\cdot\boldsymbol{Cb})\big[\boldsymbol{\tau}\cdot(\boldsymbol{B}-\boldsymbol{B}^*)\boldsymbol{\tau}\big] + 2\beta\big[(\boldsymbol{a}\cdot\boldsymbol{\tau})\big(\boldsymbol{\tau}\cdot\boldsymbol{bCB}^*\big) - (\boldsymbol{a}\cdot\boldsymbol{B\tau})(\boldsymbol{\tau}\cdot\boldsymbol{bC})\big] \\ +\alpha\|\mathbb{1}\boldsymbol{b}\|^2\cdot\|\boldsymbol{\tau}\|^2 + \beta(\boldsymbol{\tau}\cdot\boldsymbol{bC})^2 + p\|\boldsymbol{b}\|^2\big(\delta\|\boldsymbol{\tau}\|^2 + \|\boldsymbol{B\tau}\|^2\big) \\ +q\big[\delta(\boldsymbol{b}\cdot\boldsymbol{\tau})^2 + (\boldsymbol{b}\cdot\boldsymbol{B\tau})^2\big] \geq 0, \end{aligned} \tag{127}$$

which holds for any vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{\tau}$ (with $\boldsymbol{\tau}\perp\boldsymbol{n}$). We consider without loss of generality that $\boldsymbol{\tau}$ is a unitary vector, i.e. $\|\boldsymbol{\tau}\| = 1$.

Let us determine the conditions imposed on the constitutive coefficients by the Legendre-Hadamard inequality (127). Firstly, if we assume that the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel to $\boldsymbol{n}$, then the inequality (127) reduces to

$$\gamma(\boldsymbol{a}\cdot\boldsymbol{n})^2\big(\delta + \|\boldsymbol{B\tau}\|^2\big) + p(\boldsymbol{b}\cdot\boldsymbol{n})^2\big(\delta + \|\boldsymbol{B\tau}\|^2\big) \;\geq\; 0. \tag{128}$$

Because $(\boldsymbol{a}\cdot\boldsymbol{n})$ and $(\boldsymbol{b}\cdot\boldsymbol{n})$ are arbitrary, we deduce the conditions on the coefficients

$$\gamma \;\geq\; 0 \qquad \text{and} \qquad p \;\geq\; 0. \tag{129}$$

Secondly, let us consider the inequality for the tangential parts $\mathbb{1}\boldsymbol{a}$ and $\mathbb{1}\boldsymbol{b}$, following from the Legendre-Hadamard condition (127). In other words, assuming that $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors in the tangent plane $\mathcal{T}_p(\boldsymbol{x})$, the Legendre-Hadamard inequality (127) becomes

$$\begin{aligned} \alpha\|\boldsymbol{a}\|^2\big(\delta + \|\boldsymbol{B\tau}\|^2\big) + \beta\big[\delta(\boldsymbol{a}\cdot\boldsymbol{\tau})^2 + (\boldsymbol{a}\cdot\boldsymbol{B\tau})^2\big] \\ +2\alpha(\boldsymbol{a}\cdot\boldsymbol{Cb})\big[\boldsymbol{\tau}\cdot(\boldsymbol{B}-\boldsymbol{B}^*)\boldsymbol{\tau}\big] + 2\beta\big[(\boldsymbol{a}\cdot\boldsymbol{\tau})\big(\boldsymbol{\tau}\cdot\boldsymbol{bCB}^*\big) - (\boldsymbol{a}\cdot\boldsymbol{B\tau})(\boldsymbol{\tau}\cdot\boldsymbol{bC})\big] \\ +\alpha\|\boldsymbol{b}\|^2 + \beta(\boldsymbol{\tau}\cdot\boldsymbol{bC})^2 + p\|\boldsymbol{b}\|^2\big(\delta + \|\boldsymbol{B\tau}\|^2\big) + q\big[\delta(\boldsymbol{b}\cdot\boldsymbol{\tau})^2 + (\boldsymbol{b}\cdot\boldsymbol{B\tau})^2\big] \geq 0, \end{aligned} \tag{130}$$

which holds for any tangent vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{\tau}$ (with $\|\boldsymbol{\tau}\| = 1$).

If we introduce the vector $\boldsymbol{\nu} = \boldsymbol{n} \times \boldsymbol{\tau}$, then the triad $\{\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{n}\}$ is orthonormal and right-handed. We decompose the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in the basis $\{\boldsymbol{\tau}, \boldsymbol{\nu}\}$ and denote their components by

$$\boldsymbol{a} = a_\tau \boldsymbol{\tau} + a_\nu \boldsymbol{\nu} \qquad \text{and} \qquad \boldsymbol{b} = b_\tau \boldsymbol{\tau} + b_\nu \boldsymbol{\nu}. \tag{131}$$

Then, we have

$$\boldsymbol{C}\boldsymbol{b} = -\boldsymbol{b}\boldsymbol{C} = b_\nu \boldsymbol{\tau} - b_\tau \boldsymbol{\nu}, \qquad \boldsymbol{a}\cdot\boldsymbol{C}\boldsymbol{b} = a_\tau b_\nu - a_\nu b_\tau \,, \qquad \boldsymbol{\tau}\cdot\boldsymbol{b}\boldsymbol{C} = -b_\nu \,. \tag{132}$$

Also, if we designate the components of the curvature tensor $\boldsymbol{B}$ as

$$B_{\tau\tau} = \boldsymbol{\tau} \cdot \boldsymbol{B}\boldsymbol{\tau}, \qquad B_{\nu\tau} = \boldsymbol{\nu} \cdot \boldsymbol{B}\boldsymbol{\tau}, \tag{133}$$

then we can write

$$\begin{gathered} \boldsymbol{B}\boldsymbol{\tau} = B_{\tau\tau}\boldsymbol{\tau} + B_{\nu\tau}\boldsymbol{\nu}, \qquad \boldsymbol{B}^*\boldsymbol{\tau} = (2H\mathbb{1} - \boldsymbol{B})\boldsymbol{\tau} = (2H - B_{\tau\tau})\boldsymbol{\tau} - B_{\nu\tau}\boldsymbol{\nu}, \\ (\boldsymbol{B} - \boldsymbol{B}^*)\boldsymbol{\tau} = 2(B_{\tau\tau} - H)\boldsymbol{\tau} + 2B_{\nu\tau}\boldsymbol{\nu}, \\ \boldsymbol{\tau} \cdot \boldsymbol{b}\boldsymbol{C}\boldsymbol{B}^* = \boldsymbol{B}^*\boldsymbol{\tau} \cdot \boldsymbol{b}\boldsymbol{C} = (B_{\tau\tau} - 2H)b_\nu - B_{\nu\tau} b_\tau \,. \end{gathered} \tag{134}$$

Using the relations (131)-(134), we can put the inequality (130) in the equivalent form

$$\begin{gathered} \alpha(a_\tau^2 + a_\nu^2)\big(\delta + B_{\tau\tau}^2 + B_{\nu\tau}^2\big) + \beta\big[\delta\, a_\tau^2 + (B_{\tau\tau} a_\tau + B_{\nu\tau} a_\nu)^2\big] \\ +2\alpha(a_\tau b_\nu - a_\nu b_\tau)(2B_{\tau\tau} - 2H) + 2\beta\big[a_\tau\big((B_{\tau\tau} - 2H)b_\nu - B_{\nu\tau} b_\tau\big) \\ +b_\nu(B_{\tau\tau} a_\tau + B_{\nu\tau} a_\nu)\big] + \alpha(b_\tau^2 + b_\nu^2) + \beta(b_\nu^2) \\ +p(b_\tau^2 + b_\nu^2)\big(\delta + B_{\tau\tau}^2 + B_{\nu\tau}^2\big) + q\big[\delta\, b_\tau^2 + (B_{\tau\tau} b_\tau + B_{\nu\tau} b_\nu)^2\big] \geq 0, \end{gathered} \tag{135}$$

which holds for any scalars $a_\tau\,, a_\nu\,, b_\tau\,, b_\nu\,$, and for any unit vector $\boldsymbol{\tau}$ in the tangent plane.

Consider now the spectral representation of the initial curvature tensor

$$\boldsymbol{B} = \kappa_1 \boldsymbol{u}_1 \otimes \boldsymbol{u}_1 + \kappa_2 \boldsymbol{u}_2 \otimes \boldsymbol{u}_2 \,, \tag{136}$$

where $\{\boldsymbol{u}_1\,, \boldsymbol{u}_2\}$ are the orthonormal principal vectors and $\kappa_1\,, \kappa_2$ the principal curvatures of the reference midsurface at $\boldsymbol{x}$. Recall the well-known relations $\kappa_1 + \kappa_2 = 2H$ and $\kappa_1\kappa_2 = K$.

Let us choose now the tangent vector $\boldsymbol{\tau}$ to coincide with the principal vector $\boldsymbol{u}_1$, i.e. $\boldsymbol{\tau} = \boldsymbol{u}_1$. Then, we have

$$B_{\tau\tau} = \boldsymbol{u}_1 \cdot \boldsymbol{B}\boldsymbol{u}_1 = \kappa_1 \qquad \text{and} \qquad B_{\nu\tau} = \boldsymbol{u}_2 \cdot \boldsymbol{B}\boldsymbol{u}_1 = 0. \tag{137}$$

Inserting these relations into the inequality (135) we obtain the simplified form

$$\alpha(a_\tau^2+a_\nu^2)(\delta+\kappa_1^2)+\beta a_\tau^2(\delta+\kappa_1^2)+2\alpha(a_\tau b_\nu-a_\nu b_\tau)(\kappa_1-\kappa_2)+2\beta a_\tau b_\nu(\kappa_1-\kappa_2) \\ +\alpha(b_\tau^2+b_\nu^2)+\beta(b_\nu^2)+p(b_\tau^2+b_\nu^2)\big(\delta+\kappa_1^2\big)+qb_\tau^2\big(\delta+\kappa_1^2\big)\geq 0, \tag{138}$$

which holds for any real numbers a_τ, a_ν, b_τ and b_ν. Notice that (138) is a quadratic form in the variables $\{a_\tau, a_\nu, b_\tau, b_\nu\}$, which can be rearranged in the equivalent form

$$a_\tau^2(\alpha+\beta)(\delta+\kappa_1^2)+a_\nu^2\alpha\big(\delta+\kappa_1^2\big)+2a_\tau b_\nu(\alpha+\beta)(\kappa_1-\kappa_2)-2a_\nu b_\tau\alpha(\kappa_1-\kappa_2) \\ +b_\tau^2\big[\alpha+(p+q)\big(\delta+\kappa_1^2\big)\big]+b_\nu^2\big[\alpha+\beta+p\big(\delta+\kappa_1^2\big)\big]\geq 0. \tag{139}$$

The symmetric matrix of the coefficients of this quadratic form is

$$M=\begin{pmatrix} (\alpha+\beta)\big(\delta+\kappa_1^2\big) & 0 & 0 & (\alpha+\beta)(\kappa_1-\kappa_2) \\ 0 & \alpha\big(\delta+\kappa_1^2\big) & -\alpha(\kappa_1-\kappa_2) & 0 \\ 0 & -\alpha(\kappa_1-\kappa_2) & \alpha+(p+q)\big(\delta+\kappa_1^2\big) & 0 \\ (\alpha+\beta)(\kappa_1-\kappa_2) & 0 & 0 & \alpha+\beta+p\big(\delta+\kappa_1^2\big) \end{pmatrix}. \tag{140}$$

Notice that the quadratic form (139) is positive semi-definite if and only if *all* the principal minors of M are non-negative (Sylvester's criterion). Thus, the elements on the diagonal must satisfy

$$(\alpha+\beta)\big(\delta+\kappa_1^2\big)\geq 0, \qquad \alpha\big(\delta+\kappa_1^2\big)\geq 0, \\ \alpha+(p+q)\big(\delta+\kappa_1^2\big)\geq 0, \qquad \alpha+\beta+p\big(\delta+\kappa_1^2\big)\geq 0. \tag{141}$$

Remark that the factor $(\delta+\kappa_1^2)$ is always positive, since it holds $|\kappa_\alpha|h\ll 1$ and

$$\delta+\kappa_1^2=\frac{1}{h^2}\left[12-(\kappa_1 h)(\kappa_2 h)+(\kappa_1 h)^2\right]>0. \tag{142}$$

Then, in view of $p\geq 0$, the conditions on the coefficients (141) reduce to

$$\alpha\geq 0, \qquad \alpha+\beta\geq 0 \qquad \text{and} \qquad \alpha+(p+q)\big(\delta+\kappa_1^2\big)\geq 0. \tag{143}$$

Moreover, the second order principal minors of the matrix M yield the conditions

$$\begin{vmatrix} \alpha\big(\delta+\kappa_1^2\big) & -\alpha(\kappa_1-\kappa_2) \\ -\alpha(\kappa_1-\kappa_2) & \alpha+(p+q)\big(\delta+\kappa_1^2\big) \end{vmatrix}\geq 0, \qquad \text{and} \\ \begin{vmatrix} (\alpha+\beta)\big(\delta+\kappa_1^2\big) & (\alpha+\beta)(\kappa_1-\kappa_2) \\ (\alpha+\beta)(\kappa_1-\kappa_2) & \alpha+\beta+p\big(\delta+\kappa_1^2\big) \end{vmatrix}\geq 0, \tag{144}$$

which can be put in the forms

$$\alpha\Big[\alpha\big(\delta+2K-\kappa_2^2\big)+(p+q)(\delta+\kappa_1^2)^2\Big]\geq 0, \tag{145}$$

and, respectively,

$$(\alpha+\beta)\Big[(\alpha+\beta)\big(\delta+2K-\kappa_2^2\big)+p(\delta+\kappa_1^2)^2\Big]\geq 0. \tag{146}$$

Since $|\kappa_\alpha h|\ll 1$, the following factor is positive

$$\delta+2K-\kappa_2^2=\frac{1}{h^2}\left[12+(\kappa_1 h)(\kappa_2 h)-(\kappa_2 h)^2\right]>0. \tag{147}$$

Then, in view of $p\geq 0$, $\alpha\geq 0$ and $\alpha+\beta\geq 0$, we see that the inequality (146) is always satisfied, while the relation (145) reduces to

$$\alpha\big(\delta+2K-\kappa_2^2\big)+(p+q)(\delta+\kappa_1^2)^2\geq 0. \tag{148}$$

The principal minors of order 3 and 4 of the matrix M do not yield any additional restrictions on the coefficients. Thus, the constitutive coefficients α,β, p and q must satisfy the conditions (143) and (148). Analogously, if we choose the tangent vector $\boldsymbol{\tau}=\boldsymbol{u}_2$ in the inequality (135), then we obtain similar restrictions as in (137)-(148), in which the roles of κ_1 and κ_2 are interchanged. More precisely, we derive in this case the following additional conditions

$$\alpha+(p+q)\big(\delta+\kappa_2^2\big)\geq 0 \qquad \text{and} \qquad \alpha\big(\delta+2K-\kappa_1^2\big)+(p+q)(\delta+\kappa_2^2)^2\geq 0. \tag{149}$$

To resume, the Legendre-Hadamard condition (127) imposes that the constitutive coefficients α,β,γ, p and q necessarily satify the inequalities (129), (143), (148) and (149). Let us express these necessary conditions with help of the material constants λ,μ,μ_c and $a_1\,,a_2\,,a_3\,$. In view of the relations (125), the inequalities (129) and $(143)_{1,2}$ read

$$\mu+\mu_c\geq 0,\qquad \frac{\mu(\lambda+\mu)}{\lambda+2\mu}\geq 0,\qquad \frac{\mu\,\mu_c}{\mu+\mu_c}\geq 0,\qquad a_1+a_2\geq 0, \tag{150}$$

which provide the conditions

$$\mu\geq 0,\qquad \mu_c\geq 0,\qquad \frac{\lambda+\mu}{\lambda+2\mu}\geq 0 \qquad \text{and} \qquad a_1+a_2\geq 0. \tag{151}$$

Moreover, from the inequalities (143)$_3$, (148) and (149) we get the additional conditions

$$\begin{aligned}
&(\mu+\mu_c)\tfrac{h^2}{8}+(2a_1+3a_3)\big[1-\tfrac{h^2}{12}(K-\kappa_1^2)\big]\geq 0,\\
&(\mu+\mu_c)\tfrac{h^2}{8}+(2a_1+3a_3)\big[1-\tfrac{h^2}{12}(K-\kappa_2^2)\big]\geq 0,\\
&(\mu+\mu_c)\tfrac{h^2}{8}\big[1+\tfrac{h^2}{12}(K-\kappa_1^2)\big]+(2a_1+3a_3)\big[1-\tfrac{h^2}{12}(K-\kappa_2^2)\big]^2\geq 0,\\
&(\mu+\mu_c)\tfrac{h^2}{8}\big[1+\tfrac{h^2}{12}(K-\kappa_2^2)\big]+(2a_1+3a_3)\big[1-\tfrac{h^2}{12}(K-\kappa_1^2)\big]^2\geq 0.
\end{aligned}\tag{152}$$

Notice that these conditions involve also the geometrical characteristics of the reference configuration, such as the Gauß curvature K and the principal curvatures in the point $\boldsymbol{x}$. If one of these conditions is violated in any point of Ω, then the equilibrium state can be unstable. In order to avoid such instabilities of equilibrium states for shells, the inequalities (151) and (152) connecting the material constants, the thickness h and the initial curvature must hold in any point of the reference midsurface.

Remark 5. 1) Notice that the condition

$$2a_1+3a_3\geq 0 \tag{153}$$

is sufficient to ensure that the inequalities (152) hold in any point. Indeed, in view of the usual shell assumption $|\kappa_\alpha h|\ll 1$, we see that all square brackets in (152) are positive. Thus, for the sake of simpler expressions, one can replace the inequalities (152) with the more restrictive condition $2a_1+3a_3\geq 0$, which does not depend on the geometry of the shell.

2) In the case of very thin shells, the quantities $\frac{h^2}{12}K$ and $\frac{h^2}{12}\kappa_\alpha^2$ are negligible in comparison to the unity 1, i.e. we can approximate

$$1\pm\frac{h^2}{12}(K-\kappa_\alpha^2)\simeq 1. \tag{154}$$

Under this approximation, the inequalities (152) reduce to the single condition

$$(\mu+\mu_c)\frac{h^2}{8}+(2a_1+3a_3)\geq 0, \tag{155}$$

which couples the material constants μ,μ_c of the extensional part of the strain energy density with the constitutive coefficients a_1 , a_3 of the bending-curvature part of the energy.

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